MATH 590: Meshfree Methods

Chapter 11: Compactly Supported Radial Basis Functions

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Outline

As we saw earlier, compactly supported functions Φ that are truly strictly conditionally positive definite of order $m > 0$ do not exist. The compact support automatically ensures that Φ is strictly positive definite.

Another observation was that compactly supported radial functions can be strictly positive definite on $\mathbb{R}^s$ only for a fixed maximal s-value. It is not possible for a function to be strictly positive definite and radial on $\mathbb{R}^s$ for all s and also have a compact support.

Therefore we focus our attention on the characterization and construction of functions that are compactly supported, strictly positive definite and radial on $\mathbb{R}^s$ for some fixed s.

According to Bochner's theorem and generalizations thereof, a function is strictly positive definite and radial on $\mathbb{R}^s$ if its s-variate Fourier transform is non-negative.

From Appendix B:

Theorem

The Fourier transform of the radial function $\Phi = \varphi(\|\cdot\|)$ is given by another radial function

$$\hat{\Phi}(\boldsymbol{x}) = \mathcal{F}_s\varphi(\|\boldsymbol{x}\|) = \frac{1}{\sqrt{\|\boldsymbol{x}\|^{s-2}}}\int_0^\infty \varphi(t)t^{\frac{s}{2}}J_{\frac{s-2}{2}}(t\|\boldsymbol{x}\|)\mathrm{d}t,$$

where J_ν is the Bessel function of the first kind of order ν.

Remark

- *A proof of this theorem can be found in [Wendland (2005a)].*
- *This integral transform is also referred to as Fourier-Bessel transform or Hankel transform.*

Remark

- *The Hankel inversion theorem [Sneddon (1972)] ensures that the Fourier transform for radial functions is its own inverse, i.e., for radial functions φ we have*

$$\mathcal{F}_s\left[\mathcal{F}_s\varphi\right] = \varphi.$$

- *We used this earlier when we turned the Matérn functions "upside down" to get the generalized inverse multiquadrics.*

- A certain integral operator and its inverse differential operator were defined in [Schaback and Wu (1996)].
  - In that paper an entire calculus was developed for how these operators act on radial functions.
- According to [Gneiting (2002)], these operators can be traced back to [Matheron (1965)] who called the integral operator *montée* and the differential operator *descente* motivated by an application related to mining.
- In the following we define these operators and show how they facilitate the construction of compactly supported radial functions.

Definition

1. Let φ be such that $t \mapsto t\varphi(t) \in L_1[0,\infty)$. Then we define the integral operator $\mathcal{I}$ via

$$(\mathcal{I}\varphi)(r) = \int_r^\infty t\varphi(t)\mathrm{d}t, \qquad r \geq 0.$$

2. For even $\varphi \in C^2(\mathbb{R})$ we define the differential operator $\mathcal{D}$ via

$$(\mathcal{D}\varphi)(r) = -\frac{1}{r}\varphi'(r), \qquad r \geq 0.$$

In both cases the resulting functions are to be interpreted as even functions using even extensions.

Remark

Note that the integral operator $\mathcal{I}$ differs from the operator I introduced earlier by a factor t in the integrand.

The most important properties of the montée and descente operators are (see, e.g., [Schaback and Wu (1996)] or [Wendland (1995)]):

Theorem

(1) *Both $\mathcal{D}$ and $\mathcal{I}$ preserve compact support, i.e., if φ has compact support, then so do $\mathcal{D}\varphi$ and $\mathcal{I}\varphi$.*

(2) *If $\varphi \in C(\mathbb{R})$ and $t \mapsto t\phi(t) \in L_1[0,\infty)$, then $\mathcal{D}\mathcal{I}\varphi = \varphi$.*

(3) *If $\varphi \in C^2(\mathbb{R})$ ($\varphi \not\equiv 1$) is even and $\varphi' \in L_1[0,\infty)$, then $\mathcal{I}\mathcal{D}\varphi = \varphi$.*

(4) *If $t \mapsto t^{s-1}\varphi(t) \in L_1[0,\infty)$ and $s \geq 3$, then $\mathcal{F}_s(\varphi) = \mathcal{F}_{s-2}(\mathcal{I}\varphi)$.*

(5) *If $\varphi \in C^2(\mathbb{R})$ is even and $t \mapsto t^s\varphi'(t) \in L_1[0,\infty)$, then $\mathcal{F}_s(\varphi) = \mathcal{F}_{s+2}(\mathcal{D}\varphi)$.*

The operators $\mathcal{I}$ and $\mathcal{D}$ allow us to express s-variate Fourier transforms as $(s-2)$- or $(s+2)$-variate Fourier transforms, respectively.
In particular, a direct consequence of the above properties and the characterization of strictly positive definite radial functions is

Theorem

(1) *Suppose $\varphi \in C(\mathbb{R})$. If $t \mapsto t^{s-1}\varphi(t) \in L_1[0,\infty)$ and $s \geq 3$, then φ is strictly positive definite and radial on $\mathbb{R}^s$ if and only if $\mathcal{I}\varphi$ is strictly positive definite and radial on $\mathbb{R}^{s-2}$.*

(2) *If $\varphi \in C^2(\mathbb{R})$ is even and $t \mapsto t^s\varphi'(t) \in L_1[0,\infty)$, then φ is strictly positive definite and radial on $\mathbb{R}^s$ if and only if $\mathcal{D}\varphi$ is strictly positive definite and radial on $\mathbb{R}^{s+2}$.*

Remark

- *This allows us to construct new strictly positive definite radial functions from given ones by a "dimension-walk" technique that steps through multivariate Euclidean space in even increments.*
- *The examples presented in the following sections illustrate this technique.*

Probably the most popular family of compactly supported radial functions presently in use was constructed in [Wendland (1995)].

Wendland starts with the truncated power function

$$\varphi_\ell(r) = (1-r)_+^\ell.$$

which we know to be strictly positive definite and radial on $\mathbb{R}^s$ for integer $\ell \geq \lfloor \frac{s}{2} \rfloor + 1$.

Then he walks through dimensions by repeatedly applying the integral operator $\mathcal{I}$.

Definition

With $\varphi_\ell(r) = (1-r)_+^\ell$ we define

$$\varphi_{s,k} = \mathcal{I}^k \varphi_{\lfloor s/2 \rfloor + k + 1}.$$

Remark

- *Note the use of a single subscript for the truncated power function, and double subscript for the Wendland functions.*
- *It turns out that the functions $\varphi_{s,k}$ are all supported on $[0, 1]$ and have a polynomial representation there.*

More precisely,

Theorem

The functions $\varphi_{s,k}$ are strictly positive definite and radial on $\mathbb{R}^s$ and are of the form

$$\varphi_{s,k}(r) = \begin{cases} p_{s,k}(r), & r \in [0,1], \\ 0, & r > 1, \end{cases}$$

with a univariate polynomial $p_{s,k}$ of degree $\lfloor s/2 \rfloor + 3k + 1$.

Moreover, $\varphi_{s,k} \in C^{2k}(\mathbb{R})$ are unique up to a constant factor, and the polynomial degree is minimal for given space dimension s and smoothness $2k$.

Remark

- *This theorem states that any other compactly supported polynomial function that globally C^{2k} and strictly positive definite and radial on $\mathbb{R}^s$ will not have a smaller polynomial degree.*
- *Our other examples below (Wu's functions, Gneiting's functions) illustrate this fact.*
- *The strict positive definiteness of Wendland's functions $\varphi_{s,k}$ starting with non-integer values of ℓ was established in [Gneiting (1999)].*
    - *Note, however, that then the functions are no longer guaranteed to be polynomials on their support.*

Wendland gave recursive formulas for the functions $\varphi_{s,k}$ for all s, k.

We instead list the explicit formulas of [Fasshauer (1999a)].

Theorem

The functions $\varphi_{s,k}$*,* $k = 0, 1, 2, 3$*, have the form*

$$\begin{aligned}
\varphi_{s,0}(r) &= (1-r)_+^{\ell}, \\
\varphi_{s,1}(r) &\doteq (1-r)_+^{\ell+1}\left[(\ell+1)r+1\right], \\
\varphi_{s,2}(r) &\doteq (1-r)_+^{\ell+2}\left[(\ell^2+4\ell+3)r^2+(3\ell+6)r+3\right], \\
\varphi_{s,3}(r) &\doteq (1-r)_+^{\ell+3}\Big[(\ell^3+9\ell^2+23\ell+15)r^3+(6\ell^2+36\ell+45)r^2 \\
&\quad +(15\ell+45)r+15\Big],
\end{aligned}$$

where $\ell = \lfloor s/2 \rfloor + k + 1$*, and the symbol* $\doteq$ *denotes equality up to a multiplicative positive constant.*

Proof.

The case $k = 0$ follows directly from the definition.
Application of the definition for the case $k = 1$ yields

$$\begin{aligned}
\varphi_{s,1}(r) &= (\mathcal{I}\varphi_\ell)(r) = \int_r^\infty t\varphi_\ell(t)\mathrm{d}t \\
&= \int_r^\infty t(1-t)_+^\ell \mathrm{d}t \\
&= \int_r^1 t(1-t)^\ell \mathrm{d}t \\
&= \frac{1}{(\ell+1)(\ell+2)}(1-r)^{\ell+1}\left[(\ell+1)r+1\right],
\end{aligned}$$

where the compact support of φ_ℓ reduces the improper integral to a definite integral which can be evaluated using integration by parts.
The other two cases are obtained similarly by repeated application of $\mathcal{I}$. □

Example

k	$\varphi_{3,k}(r)$	smoothness
0	$(1-r)_+^2$	C^0
1	$(1-r)_+^4\,(4r+1)$	C^2
2	$(1-r)_+^6\,\left(35r^2+18r+3\right)$	C^4
3	$(1-r)_+^8\,\left(32r^3+25r^2+8r+1\right)$	C^6

Table: Wendland's compactly supported radial functions $\varphi_{s,k}$ for various choices of k and $s=3$.

Example (cont.)

- All functions in the table are strictly positive definite and radial on $\mathbb{R}^s$ for $s \leq 3$.
- Their degree of smoothness $2k$ is specified.
- The functions were determined using the direct formulas from the above theorem and thus match the definition only up to a positive constant factor.
- Note that $(x)_+^\ell$ is to be interpreted as $\left((x)_+\right)^\ell$, i.e., we first apply the cutoff function, and then the power.

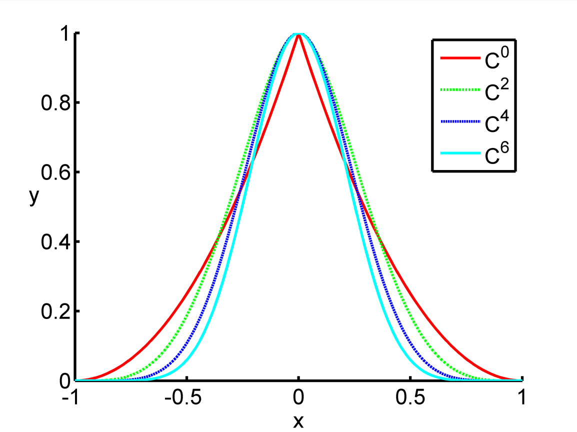


Figure: Plot of Wendland's functions $\varphi_{s,k}$ for various choices of k and $s = 3$.

For the MATLAB implementation in the next chapter it is better to express the compactly supported functions in a shifted form.
We list the appropriate functions $\widetilde{\varphi}_{s,k} = \varphi_{s,k}(1-\cdot)$ so that $\widetilde{\varphi}_{s,k}(1-\varepsilon r) = \varphi_{s,k}(\varepsilon r)$.

k	$\varphi_{3,k}(r)$	$\widetilde{\varphi}_{3,k}(r)$
0	$(1-r)_+^2$	r_+^2
1	$(1-r)_+^4\,(4r+1)$	$r_+^4\,(5-4r)$
2	$(1-r)_+^6\,\left(35r^2+18r+3\right)$	$r_+^6\,\left(56-88r+35r^2\right)$
3	$(1-r)_+^8\,\left(32r^3+25r^2+8r+1\right)$	$r_+^8\,\left(66-154r+121r^2-32r^3\right)$

Table: Wendland's compactly supported radial functions $\varphi_{s,k}$ and $\widetilde{\varphi}_{s,k} = \varphi_{s,k}(1-\cdot)$ for various choices of k and $s=3$.

In [Wu (1995b)] we find another way to construct strictly positive definite radial functions with compact support.
Wu starts with the function

$$\psi(r) = (1 - r^2)_+^\ell, \qquad \ell \in \mathbb{N},$$

which in itself is not positive definite (see the discussion at the end of Chapter 5).
However, Wu then uses convolution to construct another function that is strictly positive definite and radial on $\mathbb{R}$, i.e.,

$$\begin{aligned}
\psi_\ell(r) &= (\psi * \psi)(2r) \\
&= \int_{-\infty}^{\infty} (1 - t^2)_+^\ell (1 - (2r - t)^2)_+^\ell \mathrm{d}t \\
&= \int_{-1}^{1} (1 - t^2)^\ell (1 - (2r - t)^2)_+^\ell \mathrm{d}t.
\end{aligned}$$

Remark

This function is strictly positive definite since its Fourier transform is essentially the square of the Fourier transform of ψ and therefore non-negative.

- Just like the Wendland functions, the Wu function ψ_ℓ is a polynomial on its support.
  - The degree of the polynomial is $4\ell + 1$, and $\psi_\ell \in C^{2\ell}(\mathbb{R})$.
- Now, a family of strictly positive definite radial functions is constructed by a dimension walk using the differential operator $\mathcal{D}$.

Definition

With $\psi_\ell(r) = ((1 - \cdot^2)_+^\ell * (1 - \cdot^2)_+^\ell)(2r)$ we define

$$\psi_{k,\ell} = \mathcal{D}^k \psi_\ell.$$

The functions $\psi_{k,\ell}$

- are strictly positive definite and radial on $\mathbb{R}^s$ for $s \leq 2k+1$,
- are polynomials of degree $4\ell - 2k + 1$ on their support
- and in $C^{2(\ell-k)}$ in the interior of the support.
- On the boundary the smoothness increases to $C^{2\ell-k}$.

Example

For $\ell = 3$ we can compute the four functions

$$\psi_{k,3}(r) = \mathcal{D}^k \psi_3(r) = \mathcal{D}^k((1-\cdot^2)_+^3 * (1-\cdot^2)_+^3)(2r), \qquad k = 0, 1, 2, 3.$$

They are listed on the next slide along with their smoothness.

The maximal space dimension s for which these functions are strictly positive definite and radial on $\mathbb{R}^s$ is also listed.

Just as with the Wendland functions, the functions in Table 12 match the definition only up to a positive multiplicative constant.

Example (cont.)

k	$\psi_{k,3}(r)$	smoothness	s
0	$(1-r)_+^7(5+35r+101r^2+147r^3+101r^4+35r^5+5r^6)$	C^6	1
1	$(1-r)_+^6(6+36r+82r^2+72r^3+30r^4+5r^5)$	C^4	3
2	$(1-r)_+^5(8+40r+48r^2+25r^3+5r^4)$	C^2	5
3	$(1-r)_+^4(16+29r+20r^2+5r^3)$	C^0	7

Table: Wu's compactly supported radial functions $\psi_{k,\ell}$ for various choices of k and $\ell = 3$.

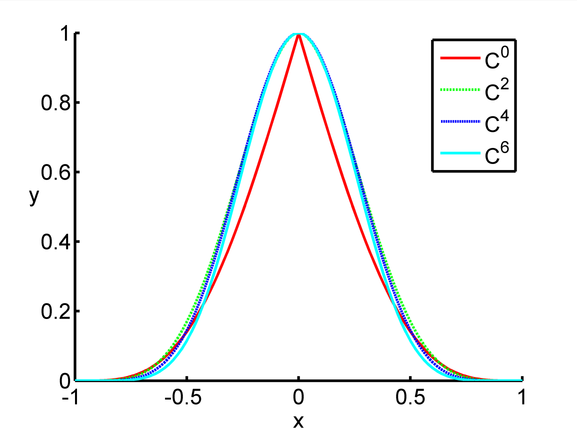


Figure: Plot of Wu's functions $\psi_{k,\ell}$ for various choices of k and $\ell = 3$.

Example (cont.)

Again, we also list the functions $\widetilde{\psi}_{k,\ell} = \psi_{k,\ell}(1-\cdot)$ used in our MATLAB implementation in Chapter 12.

k	$\widetilde{\psi}_{k,3}(r)$	smoothness	s
0	$r_+^7(429 - 1287r + 1573r^2 - 1001r^3 + 351r^4 - 65r^5 + 5r^6)$	C^6	1
1	$r_+^6(231 - 561r + 528r^2 - 242r^3 + 55r^4 - 5r^5)$	C^4	3
2	$r_+^5(126 - 231r + 153r^2 - 45r^3 + 5r^4)$	C^2	5
3	$r_+^4(70 - 84r + 35r^2 - 5r^3)$	C^0	7

Table: Shifted version $\widetilde{\psi}_{k,\ell}$ of Wu's compactly supported radial functions $\psi_{k,\ell}$ for various choices of k and $\ell = 3$.

Remark

- *As predicted by* the theorem about the Wendland functions*, for a prescribed space dimension s and smoothness the polynomial degree of Wendland's functions is lower than that of Wu's functions.*
  - *For example, both Wendland's function $\varphi_{3,2}$ and Wu's function $\psi_{1,3}$ are C^4 smooth and strictly positive definite and radial on $\mathbb{R}^3$. However, the polynomial degree of Wendland's function is 8, whereas that of Wu's function is 11.*
  - *Another comparable function is Gneiting's oscillatory function σ_2 (see below), which is a C^4 polynomial of degree 9 that is strictly positive definite and radial on $\mathbb{R}^3$.*
- *While the two families of strictly positive definite compactly supported functions discussed above are both constructed via dimension walk, Wendland uses integration (and thus obtains a family of increasingly smoother functions), whereas Wu needs to start with a function of sufficient smoothness, and then obtains successively less smooth functions (via differentiation).*

Other strictly positive definite compactly supported radial functions have been proposed by Gneiting (see, e.g., [Gneiting (2002)]).

He showed that a family of oscillatory compactly supported functions can be constructed using the so-called turning bands operator of [Matheron (1973)].

Starting with a function φ_s that is strictly positive definite and radial on $\mathbb{R}^s$ for $s \geq 3$ the turning bands operator produces

$$\varphi_{s-2}(r) = \varphi_s(r) + \frac{r\varphi_s'(r)}{s-2} \tag{1}$$

which is strictly positive definite and radial on $\mathbb{R}^{s-2}$.

Example

One such family of functions is generated if we start with the Wendland functions

$$\varphi_{s+2,1}(r) = (1-r)_+^{\ell+1}\left[(\ell+1)r+1\right]$$

(ℓ non-integer allowed).

Application of the turning bands operator results in the functions

$$\tau_{s,\ell}(r) = (1-r)_+^{\ell}\left(1+\ell r - \frac{(\ell+1)(\ell+2+s)}{s}r^2\right),$$

which are strictly positive definite and radial on $\mathbb{R}^s$ provided $\ell \geq \frac{s+5}{2}$ (see [Gneiting (2002)]).

Example (cont.)

ℓ	$\tau_{2,\ell}(r)$	smoothness
7/2	$(1-r)_+^{7/2}\left(1+\frac{7}{2}r-\frac{135}{8}r^2\right)$	C^2
5	$(1-r)_+^{5}\left(1+5r-27r^2\right)$	C^2
15/2	$(1-r)_+^{15/2}\left(1+\frac{15}{2}r-\frac{391}{8}r^2\right)$	C^2
12	$(1-r)_+^{12}\left(1+12r-104r^2\right)$	C^2

Table: Gneiting's compactly supported radial functions $\tau_{s,\ell}$ for various choices of ℓ and $s=2$.

Remark

- *All functions are in $C^2(\mathbb{R})$.*
- *If we want smoother functions, then we need to start with a smoother Wendland family as described in the next example.*

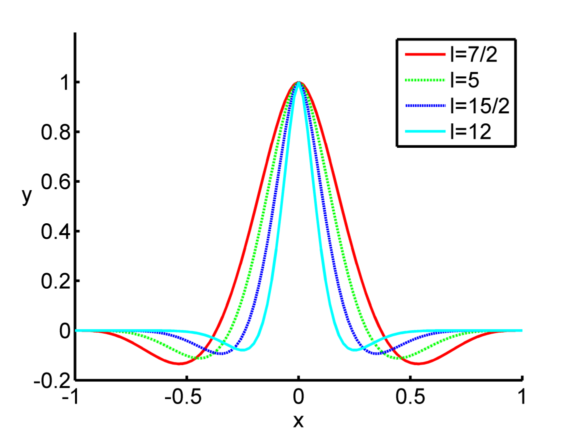


Figure: First family of oscillatory functions by Gneiting.

Example

We get a set of oscillatory functions that are strictly positive definite and radial on $\mathbb{R}^3$ by applying the turning bands operator to the Wendland functions $\varphi_{5,k}$ which are strictly positive definite and radial on $\mathbb{R}^5$ for different choices of k.
Then the resulting functions σ_k will have the same degree of smoothness $2k$ as the original functions and they will be strictly positive definite and radial on $\mathbb{R}^3$.

k	$\sigma_k(r)$	smoothness
1	$(1-r)_+^4\left(1+4r-15r^2\right)$	C^2
2	$(1-r)_+^6\left(3+18r+3r^2-192r^3\right)$	C^4
3	$(1-r)_+^8\left(15+120r+210r^2-840r^3-3465r^4\right)$	C^6

Table: Oscillatory compactly supported functions that are strictly positive definite and radial on $\mathbb{R}^3$ parametrized by smoothness.

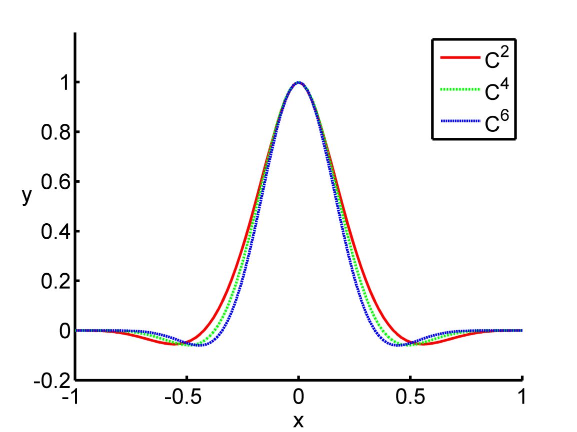


Figure: Second family of oscillatory functions by Gneiting.

Remark

Gneiting also suggests the construction of strictly positive definite radial functions by taking the product of the (appropriately scaled) Poisson functions Ω_s (see Chapter 4) with a certain compactly supported non-negative function (see [Gneiting (2002)] for more details).

Since the product of strictly positive definite functions is strictly positive definite (see Chapter 3) the resulting function will be strictly positive definite.

This will also yield oscillatory compactly supported functions.

There are many other ways in which one can construct compactly supported functions that are strictly positive definite and radial on $\mathbb{R}^s$. In [Schaback (1995a)] several such possibilities are described.

Euclid's hat functions are constructed in analogy to B-splines.

Example

It is well known that the univariate function

$$\beta(r) = (1 - |r|)_+$$

is a second-order B-spline with knots at $-1, 0, 1$, and it is obtained as the convolution of the characteristic function of the interval $[-\frac{1}{2}, \frac{1}{2}]$ with itself.

Euclid's hat functions are now obtained by convolving the characteristic function of the s-dimensional Euclidean unit ball with itself.

Euclid's hat

The resulting functions can be written for $r \in [0,1]$ in the form

$$\varphi_{2k+1}(2r) = \begin{cases} \frac{2\pi\varphi_{2k-1}(2r) - r(1-r^2)^k}{2k+1} & k = 1,2,3,\ldots, \\ 2(1-r) & k = 0, \end{cases}$$

for odd space dimensions $s = 2k+1$, and as

$$\varphi_{2k+2}(2r) = \begin{cases} \frac{2\pi\varphi_{2k}(2r) - r\sqrt{(1-r^2)}(1-r^2)^k}{2k+2} & k = 1,2,3,\ldots, \\ 2(\arccos r - r\sqrt{1-r^2}) & k = 0, \end{cases}$$

for even space dimensions $s = 2k$.

Note that these functions are defined to be zero outside the interval $[0,2]$.

Euclid's hat (cont.)

In the table below we have employed a substitution $2r \to r$ and a normalization factor such that the functions all have a value of one at the origin.

s	$\varphi_s(r)$	smoothness
1	$1 - \frac{r}{2}$	C^0
2	$\frac{1}{2\pi}\left(4\arccos\left(\frac{r}{2}\right) - r\sqrt{4-r^2}\right)$	C^0
3	$1 - \frac{1}{32\pi}\left((4+16\pi)r - r^3\right)$	C^0
4	$\frac{2}{\pi}\arccos\left(\frac{r}{2}\right) - \frac{1}{32\pi}\sqrt{4-r^2}\left(20r + r^3\right)$	C^0
5	$1 - \frac{1}{64\pi^2}\left((12+8\pi+32\pi^2)r - (3+2\pi)r^3\right)$	C^0

Table: Euclid's hat functions (defined for $0 \le r \le 2$) for different values of s.

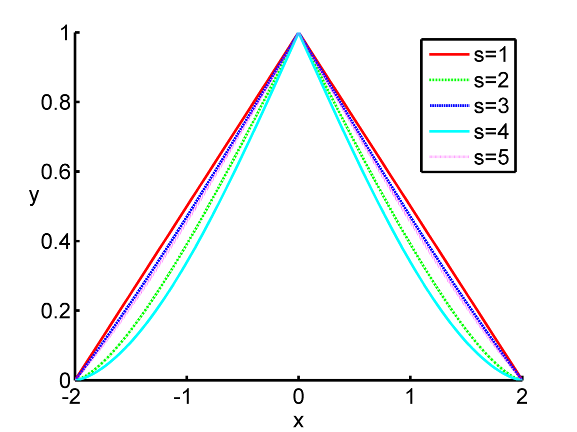


Figure: Euclid's hat functions.

Remark

- *Another construction described in [Schaback (1995a)] is the radialization of the s-fold tensor product of univariate B-splines of even order $2m$ with uniform knots.*
- *These functions do not seem to have a simple representation that lends itself to numerical computations.*
- *As can be seen from its radialized Fourier transform, the radialized B-spline itself is not strictly positive definite and radial on any $\mathbb{R}^s$ with $s > 1$.*
- *For $s = 1$ only the B-splines of even order are strictly positive definite (see, e.g., [Schölkopf and Smola (2002)]).*

The last family of compactly supported strictly positive definite radial functions we would like to mention is due to [Buhmann (1998)].

Buhmann's functions contain a logarithmic term in addition to a polynomial.

His functions have the general form

$$\varphi(r) = \int_0^\infty \left(1 - \frac{r^2}{t}\right)_+^\lambda t^\alpha (1 - t^\delta)_+^\rho \mathrm{d}t.$$

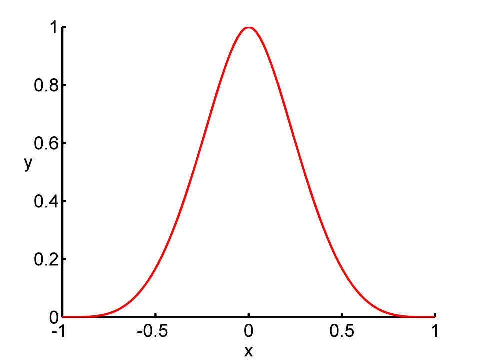
Here $0 < \delta \leq \frac{1}{2}$, $\rho \geq 1$, and in order to obtain functions that are strictly positive definite and radial on $\mathbb{R}^s$ for $s \leq 3$ the constraints for the remaining parameters are $\lambda \geq 0$, and $-1 < \alpha \leq \frac{\lambda-1}{2}$.

Example

Let $\alpha = \delta = \frac{1}{2}$, $\rho = 1$ and $\lambda = 2$ (see [Buhmann (2000)]):

$$\varphi(r) \doteq 12r^4 \log r - 21r^4 + 32r^3 - 12r^2 + 1, \qquad 0 \le r \le 1.$$

This function is in $C^2(\mathbb{R})$ and strictly positive definite and radial on $\mathbb{R}^s$ for $s \le 3$.



Remark

- *In [Buhmann (2000)] it is stated that his construction encompasses both Wendland's and Wu's functions.*
- *An even more general theorem that shows that integration of a positive function $f \in L_1[0, \infty)$ against a strictly positive definite kernel K results in a strictly positive definite function can be found in [Wendland (2005a)] (see also Chapter 4).*
  - *More specifically,*
    $$\varphi(r) = \int_0^\infty K(t, r) f(t) \mathrm{d}t$$
    is strictly positive definite.
  - *Buhmann's construction then corresponds to choosing*
    $f(t) = t^\alpha (1 - t^\delta)_+^\rho$ *and* $K(t, r) = \left(1 - \frac{r^2}{t}\right)_+^\lambda$.

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