MATH 461: Fourier Series and Boundary Value Problems

Chapter VII: Higher-Dimensional PDEs

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Outline

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2. PDEs in Space
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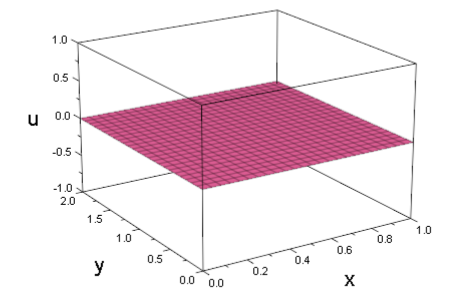
We now derive a generalization of the wave equation to two dimensions (see Chapter 4.5 of [Haberman]).

Consider a stretched elastic membrane of unspecified shape (e.g., circular or rectangular) with equilibrium position in the xy-plane.

Every point $(x, y, 0)$ of the membrane has a displacement $z = u(x, y, t)$ at time t.



As for the vibrating string we assume:

- There are only small vertical displacements.
- The membrane is perfectly flexible.

In addition we make the simplifying assumptions:

- The tensile force is constant.
- There are no external forces acting on the membrane.

As a consequence of these assumptions the tensile force $\mathbf{F}_T$ will be tangential to the membrane acting along the entire boundary of the membrane, i.e.,

$$\mathbf{F}_T = T_0 \left(\hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}} \right),$$

where

- T_0 is the constant tension,
- $\hat{\boldsymbol{t}}$ is the unit tangent vector along the edge of the membrane,
- $\hat{\boldsymbol{n}}$ is the unit outer surface normal to the membrane.

As with the string, we need only the vertical component of the tensile force, i.e.,

$$T_v = \mathbf{F}_T \cdot \hat{\boldsymbol{k}} = T_0 \left(\hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}} \right) \cdot \hat{\boldsymbol{k}},$$

where $\hat{\boldsymbol{k}}$ is the standard unit vector $(0, 0, 1)$.
Note that $\mathbf{F}_T, \hat{\boldsymbol{t}}, \hat{\boldsymbol{n}}$ and T_v are all functions of x, y and t.

As with the vibrating string we use Newton's law, $F = m\,a$, with

- mass $m = \rho_0\,\mathrm{d}A$, where ρ_0 is the density, and $\mathrm{d}A$ is the surface area element, and
- acceleration $a = \frac{\partial^2 u}{\partial t^2}$.

The balance of forces equation now reads

$$\iint\limits_R \rho_0 \frac{\partial^2 u}{\partial t^2}\,\mathrm{d}A = \int\limits_{\partial R} T_0 \left(\hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}}\right) \cdot \hat{\boldsymbol{k}}\,\mathrm{d}s \tag{1}$$

with arc length element $\mathrm{d}s$.

In order to obtain a PDE we need to convert the boundary integral on the right-hand side of (1) to a surface integral.

Stokes' theorem[1] tells us

$$\int_{\partial R} \mathbf{F} \cdot \hat{\boldsymbol{t}} \, \mathrm{d}s = \iint_R (\nabla \times \mathbf{F}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}A,$$

i.e., the boundary integral of the tangential component of the vector field **F** is equal to the surface integral of the normal component of the curl of **F**.

However, our boundary integral

$$\int_{\partial R} T_0 \left(\hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}} \right) \cdot \hat{\boldsymbol{k}} \, \mathrm{d}s$$

does not match the form needed for Stokes, so we first need to work on this integral.

[1] Recall that Stokes' theorem is a variant of Green's theorem (2D divergence theorem) applicable to non-planar regions

The vector triple product

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = (\boldsymbol{b} \times \boldsymbol{c}) \cdot \boldsymbol{a} = (\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{b}$$

allows us to rewrite

$$\int_{\partial R} T_0 \left(\hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}}\right) \cdot \hat{\boldsymbol{k}} \, \mathrm{d}s = \int_{\partial R} T_0 \left(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{k}}\right) \cdot \hat{\boldsymbol{t}} \, \mathrm{d}s$$

which now has the tangential component of a vector field as its integrand, so that it matches Stokes.
Therefore, using Stokes' theorem, we have

$$\int_{\partial R} T_0 \left(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{k}}\right) \cdot \hat{\boldsymbol{t}} \, \mathrm{d}s = \iint_R T_0 \left[\nabla \times \left(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{k}}\right)\right] \cdot \hat{\boldsymbol{n}} \, \mathrm{d}A, \tag{2}$$

and we can now return to (1).

Replacing the right-hand side of (1) by the right-hand side of (2) we have

$$\iint_R \rho_0 \frac{\partial^2 u}{\partial t^2} \, \mathrm{d}A = \iint_R T_0 \left[\nabla \times \left(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{k}}\right)\right] \cdot \hat{\boldsymbol{n}} \, \mathrm{d}A.$$

Since this identity holds for any region R we must have

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \left[\nabla \times \left(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{k}}\right)\right] \cdot \hat{\boldsymbol{n}}. \tag{3}$$

The problem with this equation is that there is no displacement u on the right-hand side.

Where does u enter the right-hand side $T_0 \left[\nabla \times \left(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{k}}\right)\right] \cdot \hat{\boldsymbol{n}}$?

Through the normal vector $\hat{\boldsymbol{n}}$.

Treating the membrane $z = u(x, y)$ as a level surface

$$f(x, y, z) = 0 \quad \Longleftrightarrow \quad u(x, y) - z = 0$$

we know that the normal vector is parallel to the gradient of f, i.e.,

$$\hat{\boldsymbol{n}} = \frac{-\frac{\partial u}{\partial x}\hat{\boldsymbol{i}} - \frac{\partial u}{\partial y}\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + 1}} \approx -\frac{\partial u}{\partial x}\hat{\boldsymbol{i}} - \frac{\partial u}{\partial y}\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$$

if we have small displacements, i.e., $\left(\frac{\partial u}{\partial x}\right)^2$ and $\left(\frac{\partial u}{\partial y}\right)^2$ are small.

Then

$$\hat{\boldsymbol{n}} \times \hat{\boldsymbol{k}} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ -\frac{\partial u}{\partial x} & -\frac{\partial u}{\partial y} & 1 \\ 0 & 0 & 1 \end{vmatrix} = -\frac{\partial u}{\partial y}\hat{\boldsymbol{i}} + \frac{\partial u}{\partial x}\hat{\boldsymbol{j}}$$

and (since $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ don't depend on z)

$$\nabla \times \left(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{k}}\right) = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} & 0 \end{vmatrix} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\hat{\boldsymbol{k}}.$$

Finally, using the previous result and since we are using $\hat{\boldsymbol{n}} = -\frac{\partial u}{\partial x}\hat{\boldsymbol{i}} - \frac{\partial u}{\partial y}\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$, which implies $\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}} = 1$, we have

$$\left[\nabla \times \left(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{k}}\right)\right] \cdot \hat{\boldsymbol{n}} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

and so we get from (3)

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

or

$$\frac{\partial^2 u}{\partial t^2}(x, y, t) = c^2 \nabla^2 u(x, y, t),$$

where $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ is the (spatial) Laplacian and $c^2 = \frac{T_0}{\rho_0}$.

This is the standard form of the wave equation in 2D.

Remark

The steady-state problem, i.e., $\frac{\partial^2 u}{\partial t^2} = 0$, leads to

$$\nabla^2 u(x, y) = 0 \qquad \text{(Laplace's equation).}$$

If an external force is added to the steady-state problem, then we get

$$\nabla^2 u(x, y) = f(x, y) \qquad \text{(Poisson's equation).}$$

So far we used separation of variables only for PDEs with two independent variables, such as $u(x,t)$, $u(x,y)$, or $u(r,\theta)$.

Now we will consider PDEs in space, i.e., we will have to deal with

- functions of three variables such as $u(x,y,t)$, $u(x,y,z)$, or $u(r,\theta,t)$,
- or even functions of four variables such as $u(x,y,z,t)$ or $u(\rho,\varphi,\theta,t)$.

Corresponding PDEs might be

- a 2D or 3D heat equation (in Cartesian or in polar coordinates)
$$\frac{\partial u}{\partial t} = k\nabla^2 u,$$
- a 2D or 3D wave equation (in Cartesian or in polar coordinates)
$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u,$$
- a steady-state 3D heat or wave equation
$$\nabla^2 u = 0.$$

We will now look at two examples and see how to apply separation of variables in these different cases:

- vibrations of an arbitrarily shaped membrane, i.e., a 2D wave equation,
- heat conduction in an arbitrary solid, i.e., a 3D heat equation,

We will see that we can separate time from space and then obtain

- one of our usual ODEs for the time problem,
- but a PDE eigenvalue problem for space.

Vibrations of an arbitrarily shaped membrane

Let's consider the PDE

$$\frac{\partial^2 u}{\partial t^2}(x, y, t) = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)(x, y, t),$$

a 2D wave equation, with initial conditions

$$\begin{aligned} u(x, y, 0) &= f(x, y) \qquad \text{(initial displacement)} \\ \frac{\partial u}{\partial t}(x, y, 0) &= g(x, y) \qquad \text{(initial velocity)} \end{aligned}$$

We cannot specify any boundary conditions at this point since the shape of the domain is not given.

For separation of variables we start with the *Ansatz*

$$u(x,y,t) = T(t)\varphi(x,y)$$

so that the partial derivatives are

$$\frac{\partial^2 u}{\partial t^2}(x,y,t) = T''(t)\varphi(x,y),$$
$$\frac{\partial^2 u}{\partial x^2}(x,y,t) = T(t)\frac{\partial^2 \varphi}{\partial x^2}(x,y), \quad \frac{\partial^2 u}{\partial y^2}(x,y,t) = T(t)\frac{\partial^2 \varphi}{\partial y^2}(x,y),$$

and the wave equation turns into

$$T''(t)\varphi(x,y) = c^2 T(t)\left(\frac{\partial^2 \varphi}{\partial x^2}(x,y) + \frac{\partial^2 \varphi}{\partial y^2}(x,y)\right)$$

or

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{\frac{\partial^2 \varphi}{\partial x^2}(x,y) + \frac{\partial^2 \varphi}{\partial y^2}(x,y)}{\varphi(x,y)} = -\lambda.$$

As a result we have

- one well-known ODE for time:

$$T''(t) = -\lambda c^2 T(t),$$

  which has oscillatory solutions for $\lambda > 0$, and

- one PDE for the spatial part:

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial x^2}(x,y) + \frac{\partial^2 \varphi}{\partial y^2}(x,y) &= -\lambda \varphi(x,y) \\ \iff \qquad \nabla^2 \varphi(x,y) &= -\lambda \varphi(x,y). \end{aligned}$$

  This PDE eigenvalue equation is known as the Helmholtz equation.
  We will look at more detailed examples later.

Remark

In order to attempt a solution of the Helmholtz equation (with the help of separation of variables) we will need to have a "nice" region and appropriate boundary conditions.

- *If the region is rectangular, then we can separate*

$$\varphi(x, y) = X(x)Y(y).$$

- *If the region is circular, then*

$$\varphi(x, y) = \tilde{\varphi}(r, \theta) = R(r)\Theta(\theta)$$

will work.

Heat conduction in an arbitrary solid

Now we consider the PDE

$$\frac{\partial u}{\partial t}(x,y,z,t) = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)(x,y,z,t),$$

a 3D heat equation, with initial temperature

$$u(x,y,z,0) = f(x,y,z).$$

Again, we cannot specify any boundary conditions at this point since the shape of the domain is not given.

For separation of variables we start with the *Ansatz*

$$u(x,y,z,t) = T(t)\varphi(x,y,z)$$

and have the partial derivatives

$$\begin{aligned}\frac{\partial u}{\partial t}(x,y,z,t) &= T'(t)\varphi(x,y,z),\\ \left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)(x,y,z,t) &= T(t)\left(\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}+\frac{\partial^2 \varphi}{\partial z^2}\right)(x,y,z)\end{aligned}$$

so that the heat equation turns into

$$T'(t)\varphi(x,y,z) = kT(t)\left(\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}+\frac{\partial^2 \varphi}{\partial z^2}\right)(x,y,z)$$

or

$$\frac{1}{k}\frac{T'(t)}{T(t)} = \frac{\nabla^2\varphi(x,y,z)}{\varphi(x,y,z)} = -\lambda.$$

For this example we get

- the well-known time ODE

$$T'(t) = -\lambda k T(t),$$

  with solution for $T(t) = e^{-\lambda k t}$, and
- once again the Helmholtz PDE for the spatial part:

$$\begin{aligned}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\right)(x,y,z) &= -\lambda\varphi(x,y,z) \\ \iff \qquad \nabla^2\varphi(x,y,z) &= -\lambda\varphi(x,y,z).\end{aligned}$$

Let's assume the membrane has dimensions $0 \le x \le L$ and $0 \le y \le H$.
The wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

and we will consider Dirichlet boundary conditions

$$u(0, y, t) = u(L, y, t) = u(x, 0, t) = u(x, H, t) = 0$$

along with the standard initial conditions

$$\begin{aligned} u(x, y, 0) &= f(x, y) \\ \frac{\partial u}{\partial t}(x, y, 0) &= g(x, y). \end{aligned}$$

Separation of variables with *Ansatz* $u(x,y,t) = T(t)\varphi(x,y)$ results in the ODE

$$T''(t) = -\lambda c^2 T(t)$$

and the Helmholtz PDE eigenvalue problem

$$\frac{\partial^2 \varphi}{\partial x^2}(x,y) + \frac{\partial^2 \varphi}{\partial y^2}(x,y) = -\lambda \varphi(x,y)$$

with boundary conditions

$$\varphi(0,y) = \varphi(L,y) = \varphi(x,0) = \varphi(x,H) = 0.$$

We can now investigate the solution of this eigenvalue problem by another separation of variables *Ansatz* (chances are good this will work since the PDE and BCs are linear and homogeneous).

We let

$$\varphi(x, y) = X(x)Y(y)$$

so that $\frac{\partial^2\varphi}{\partial x^2}(x, y) = X''(x)Y(y)$ and $\frac{\partial^2\varphi}{\partial y^2}(x, y) = X(x)Y''(y)$.

Then the Helmholtz equation becomes

$$X''(x)Y(y) + X(x)Y''(y) = -\lambda X(x)Y(y)$$

or

$$\frac{X''(x)}{X(x)} = -\lambda - \frac{Y''(y)}{Y(y)} = -\mu$$

with a new separation constant μ.

As a result, we now have two Sturm–Liouville eigenvalue problems:

- The well-known problem

$$X''(x) = -\mu X(x)$$
$$\text{with BCs} \quad X(0) = X(L) = 0$$

which yields eigenvalues and eigenfunctions

$$\mu_n = \left(\frac{n\pi}{L}\right)^2, \qquad X_n(x) = \sin\frac{n\pi x}{L}, \qquad n = 1, 2, 3, \ldots$$

- and the set of slightly modified problems (each one corresponding to one of the solutions of the first problem)

$$Y''(y) = -(\lambda - \mu_n)Y(y), \qquad n = 1, 2, 3 \ldots$$
$$\text{with BCs} \quad Y(0) = Y(H) = 0.$$

Here we get the eigenvalues and eigenfunctions

$$\lambda_{n,m} - \mu_n = \left(\frac{m\pi}{H}\right)^2, \quad Y_{n,m}(y) = \sin\frac{m\pi y}{H}, \quad n, m = 1, 2, 3, \ldots$$

Inserting the eigenvalues $\mu_n = \left(\frac{n\pi}{L}\right)^2$ into the expression for the eigenvalues $\lambda_{n,m}$ of the second problem we get

$$\lambda_{n,m} = \left(\frac{m\pi}{H}\right)^2 + \mu_n = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2, \qquad n, m = 1, 2, 3, \ldots$$

Since we assumed $\varphi(x, y) = X(x)Y(y)$ we have the combined eigenfunctions

$$\varphi_{n,m}(x, y) = X_n(x)Y_{n,m}(y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}, \qquad n, m = 1, 2, 3, \ldots$$

Using the eigenvalues $\lambda_{n,m}$ in the time ODE $T''(t) = -\lambda c^2 T(t)$ we have (note that all eigenvalues are positive)

$$T_{n,m}(t) = c_1 \cos\sqrt{\lambda_{n,m}}ct + c_2 \sin\sqrt{\lambda_{n,m}}ct.$$

By the principle of superposition we get the general solution of the vibrating membrane problem (before using the ICs) as

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[a_{n,m}\cos\sqrt{\lambda_{n,m}}ct + b_{n,m}\sin\sqrt{\lambda_{n,m}}ct\right]\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}.$$

This is a double Fourier sine series, and we find the coefficients using the initial conditions:

$$u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{n,m}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H} \stackrel{!}{=} f(x,y).$$

Here we can interpret, holding x fixed,

$$\sum_{n=1}^{\infty} a_{n,m}\sin\frac{n\pi x}{L}$$

as the Fourier sine coefficient of the function $y \mapsto f(x,y)$, i.e.,

$$\sum_{n=1}^{\infty} a_{n,m}\sin\frac{n\pi x}{L} = \frac{2}{H}\int_0^H f(x,y)\sin\frac{m\pi y}{H}dy, \qquad m = 1,2,3,\ldots \tag{4}$$

Now we note that the right-hand side of (4) is itself some function of x, i.e.,

$$F(x) = \frac{2}{H}\int_0^H f(x,y)\sin\frac{m\pi y}{H}\mathrm{d}y, \tag{5}$$

and so (4) can be interpreted as

$$F(x) = \sum_{n=1}^{\infty} a_{n,m}\sin\frac{n\pi x}{L}, \qquad m = 1,2,3,\ldots,$$

which gives us $a_{n,m}$ as Fourier sine coefficients of F, i.e.,

$$\begin{aligned} a_{n,m} &= \frac{2}{L}\int_0^L F(x)\sin\frac{n\pi x}{L}\mathrm{d}x \\ &\overset{(5)}{=} \frac{2}{L}\int_0^L \left[\frac{2}{H}\int_0^H f(x,y)\sin\frac{m\pi y}{H}\mathrm{d}y\right]\sin\frac{n\pi x}{L}\mathrm{d}x \end{aligned}$$

We therefore have

$$a_{n,m} = \frac{2}{L}\frac{2}{H}\int_0^L \left[\int_0^H f(x,y)\sin\frac{m\pi y}{H}\mathrm{d}y\right]\sin\frac{n\pi x}{L}\mathrm{d}x, \quad n,m = 1,2,3,\ldots$$

To find the coefficients $b_{n,m}$ we need the t-partial of the general solution u:

$$\begin{aligned}\frac{\partial u}{\partial t}(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[-\sqrt{\lambda_{n,m}}ca_{n,m}\sin\sqrt{\lambda_{n,m}}ct + \sqrt{\lambda_{n,m}}cb_{n,m}\cos\sqrt{\lambda_{n,m}}ct\right] \\ &\qquad \times \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\end{aligned}$$

so that

$$\frac{\partial u}{\partial t}(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sqrt{\lambda_{n,m}}cb_{n,m}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H} \stackrel{!}{=} g(x,y).$$

Following the same procedure as before, we first get the Fourier sine coefficients of the function $y \mapsto g(x,y)$ (i.e., x is held fixed) as

$$\sum_{n=1}^{\infty} \sqrt{\lambda_{n,m}} c b_{n,m} \sin \frac{n\pi x}{L} = \frac{2}{H} \int_0^H g(x,y) \sin \frac{m\pi y}{H} dy, \quad m = 1,2,3,\ldots$$
$$= G(x),$$

and then $\sqrt{\lambda_{n,m}} c b_{n,m}$ as the Fourier sine coefficients of G, i.e.,

$$b_{n,m} = \frac{1}{c\sqrt{\lambda_{n,m}}} \frac{2}{L} \frac{2}{H} \int_0^L \left[\int_0^H g(x,y) \sin \frac{m\pi y}{H} dy \right] \sin \frac{n\pi x}{L} dx,$$
$$n, m = 1,2,3,\ldots$$

Remark

There are other (equivalent) ways in which we could have approached this problem.

- *For example, the order in which we find the eigenfunctions X_n and $Y_{n,m}$ does not matter. However, if we reversed the order, we would be enumerating them as Y_n and $X_{n,m}$.*
- *We also could have made a 3-way separation of variables right off the bat. This is described in Appendix 7.3 in [Haberman].*

In analogy to the 1D Sturm–Liouville equation $\varphi''(x) + \lambda\varphi(x) = 0$ we now investigate the Helmholtz equation

$$\nabla^2\varphi + \lambda\varphi = 0$$

subject to a boundary condition of the form

$$a\varphi + b\nabla\varphi \cdot \hat{\boldsymbol{n}} = 0,$$

where a and b are both functions of x and y, the coordinates of points on the boundary, and $\varphi \cdot \hat{\boldsymbol{n}}$ is the normal derivative of φ along the boundary.

More generally, we could even consider a Sturm–Liouville-type equation of the form

$$\nabla \cdot (p\nabla\varphi) + q\varphi + \lambda\sigma\varphi = 0$$

with coefficient functions p, q and σ.

The Helmholtz equation corresponds to $p \equiv 1$, $q \equiv 0$ and $\sigma \equiv 1$.

Properties of the 2D Helmholtz equation

- Analytic solutions of the Helmholtz eigenvalue problem are known only for simple geometries such as rectangles, triangles or circles.
- For more complicated domains one needs to use numerical methods such as finite elements.
- However, one can still prove qualitative results.

We illustrate these properties with the help of

$$\nabla^2\varphi + \lambda\varphi = 0, \qquad 0 < x < L,\ 0 < y < H$$
$$\varphi = 0 \quad \text{on the boundary of } [0, L] \times [0, H]$$

with its eigenvalues and eigenfunctions

$$\begin{aligned} \lambda_{n,m} &= \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2, \quad n, m = 1, 2, 3, \ldots \\ \varphi_{n,m}(x, y) &= \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}. \end{aligned}$$

Similar to regular 1D Sturm–Liouville problems we have:

1. All eigenvalues are real, i.e., we do not need to search for complex eigenvalues.
   This is obvious for the example problem since
   $$\lambda_{n,m} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2, \quad n, m = 1, 2, 3, \ldots$$
2. There are infinitely many eigenvalues that can be ordered (but no longer strictly).
   For the example problem
   $$\lambda_{1,1} = \left(\frac{\pi}{L}\right)^2 + \left(\frac{\pi}{H}\right)^2$$
   is the smallest one. However, the rest of the ordering depends on L and H.
   There is no largest eigenvalue.

3. There may be more than one eigenfunction associated with any eigenvalue.

This suggests that there can be different modes (eigenfunctions) that vibrate with the same frequency (eigenvalue).

This property is different from the 1D case.

Example

Choose $L = 2H$ in our example problem. Then

$$\lambda_{n,m} = \frac{n^2\pi^2}{4H^2} + \frac{m^2\pi^2}{H^2} = \frac{\pi^2}{4H^2}\left(n^2 + 4m^2\right)$$

and

$$\varphi_{n,m}(x,y) = \sin\frac{n\pi x}{2H}\sin\frac{m\pi y}{H}.$$

Now, note that

$$\lambda_{4,1} = \frac{\pi^2}{4H^2}\left(4^2 + 4\cdot 1^2\right) = \frac{5\pi^2}{H^2} = \frac{\pi^2}{4H^2}\left(2^2 + 4\cdot 2^2\right) = \lambda_{2,2}$$

so that

$$\begin{aligned}\varphi_{4,1}(x,y) &= \sin\frac{4\pi x}{2H}\sin\frac{\pi y}{H} = \sin\frac{2\pi x}{H}\sin\frac{\pi y}{H}\\ \varphi_{2,2}(x,y) &= \sin\frac{2\pi x}{2H}\sin\frac{2\pi y}{H} = \sin\frac{\pi x}{H}\sin\frac{2\pi y}{H}\end{aligned}$$

and we have two different eigenfunctions associated with the same (double, i.e., not strictly ordered) eigenvalue.

Remark

Eigenvalues can also have multiplicities higher than two.

Again, for the example $L = 2H$ we have, e.g.,

$$\lambda_{2,8} = \lambda_{8,7} = \lambda_{14,4} = \lambda_{16,1} = \frac{65\pi^2}{H^2}.$$

4. The set of eigenfunctions $\{\varphi_{n,m}\}_{n,m=1}^{\infty}$ is complete, i.e., any piecewise smooth function f can be represented by a generalized Fourier series

$$f(x,y) \sim \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{n,m}\varphi_{n,m}(x,y)$$

In our example

$$f(x,y) \sim \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{n,m}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}.$$

5. Eigenfunctions associated with different eigenvalues are orthogonal on the region R with respect to the weight $\sigma \equiv 1$, i.e.,

$$\iint_R \varphi_{\lambda_1}(x,y)\varphi_{\lambda_2}(x,y)\,\mathrm{d}A = 0 \quad \text{if } \lambda_1 \neq \lambda_2.$$

In our example, provided $\lambda_{n_1,m_1} \neq \lambda_{n_2,m_2}$,

$$\int_0^L \int_0^H \left(\sin\frac{n_1\pi x}{L}\sin\frac{m_1\pi y}{H}\right)\left(\sin\frac{n_2\pi x}{L}\sin\frac{m_2\pi y}{H}\right)\mathrm{d}y\mathrm{d}x = 0$$

and the Fourier coefficients are

$$a_{n,m} = \frac{\int_0^L \int_0^H f(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\,\mathrm{d}y\mathrm{d}x}{\int_0^L \int_0^H \sin^2\frac{n\pi x}{L}\sin^2\frac{m\pi y}{H}\,\mathrm{d}y\mathrm{d}x}.$$

6. The Rayleigh quotient can be formed and used as in 1D. In particular,

$$\lambda = \frac{-\displaystyle\int_{\partial R} \varphi \nabla\varphi \cdot \hat{\boldsymbol{n}} \mathrm{d}s + \iint_R |\nabla\varphi|^2 \,\mathrm{d}A}{\displaystyle\iint_R \varphi^2 \mathrm{d}A}$$

7. The convergence properties are as in Chapter 5.10, i.e., the mean square error

$$\iint_R \left[f(x,y) - \sum_\lambda a_\lambda \varphi_\lambda(x,y) \right]^2 dxdy,$$

where the number of terms in the sum $\sum_\lambda$ is finite, is minimized for $\alpha_\lambda = a_\lambda$, the generalized Fourier coefficients of f.

In 1D we had Green's formula

$$\int_a^b [u(x)(\mathcal{L}v)(x) - v(x)(\mathcal{L}u)(x)]\, \mathrm{d}x = \left[p(x)\left(u(x)v'(x) - v(x)u'(x)\right)\right]_a^b,$$

where $\mathcal{L}u = \frac{\mathrm{d}}{\mathrm{d}x}(pu') + qu$ stood for the Sturm–Liouville operator.

The self-adjointness of $\mathcal{L}$ was characterized by

$$\int_a^b [u(x)(\mathcal{L}v)(x) - v(x)(\mathcal{L}u)(x)]\, \mathrm{d}x = 0.$$

Now we will state analogous results for the 2D operator $\mathcal{L}u = \nabla^2 u$.

In this case, Green's formula is obtained with the help of Green's theorem and the identity (analogous to the product rule)

$$\nabla \cdot (u\nabla v) = \nabla u \cdot \nabla v + u\nabla^2 v \quad (6)$$

$$\iint_R \left[u(\nabla^2 v) - v(\nabla^2 u)\right] \, dA \overset{(6)}{=} \iint_R \nabla \cdot [u\nabla v - v\nabla u] \, dA$$

$$\overset{\text{Green's Thm}}{=} \int_{\partial R} (u\nabla v - v\nabla u) \cdot \hat{\boldsymbol{n}} ds$$

Here we have the vector field $\mathbf{F} = u\nabla v - v\nabla u$, so that $\nabla \cdot [u\nabla v - v\nabla u] = \text{div}\mathbf{F}$ and the boundary integral has the normal component of $\mathbf{F}$ as its integrand.

Remark

- *Green's formula is known in Calc III as Green's second identity.*
- *If the BCs are such that u and v (or $\nabla u \cdot \hat{\mathbf{n}}$ and $\nabla v \cdot \hat{\mathbf{n}}$) are zero on the boundary, ∂R, then $\mathcal{L} = \nabla^2$ will be self-adjoint.*

To investigate the vibrations of a circular drum we need to use the wave equation in polar coordinates, i.e.,

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c^2 \nabla^2 u \\ &= c^2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right], \quad 0 < r < a, \; -\pi < \theta < \pi.\end{aligned}$$

The only boundary condition we have is

$$u(a, \theta, t) = 0, \qquad -\pi < \theta < \pi, \quad t > 0,$$

and the initial conditions are the standard ones

$$\begin{aligned} u(r, \theta, 0) &= f(r, \theta) \\ \frac{\partial u}{\partial t}(r, \theta, 0) &= g(r, \theta). \end{aligned}$$

We begin with a separation of variables *Ansatz* (just like in the section for the rectangular drum) $u(r,\theta,t) = \varphi(r,\theta)T(t)$ so that we get the ODE

$$T''(t) = -\lambda c^2 T(t)$$

and the Helmholtz PDE (in polar coordinates)

$$\nabla^2\varphi + \lambda\varphi = 0$$
$$\text{with BC} \quad \varphi(a,\theta) = 0.$$

We can write this PDE eigenvalue problem as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \lambda\varphi = 0$$
$$\varphi(a,\theta) = 0.$$

Now we again apply separation of variables for this polar coordinate problem (as we did in Chapter 2) using the *Ansatz* $\varphi(r,\theta) = R(r)\Theta(\theta)$. This gives us

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left[R(r)\Theta(\theta)\right]\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left[R(r)\Theta(\theta)\right] + \lambda\left[R(r)\Theta(\theta)\right] = 0$$

or

$$\frac{\Theta(\theta)}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rR'(r)\right) + \frac{R(r)}{r^2}\Theta''(\theta) + \lambda R(r)\Theta(\theta) = 0.$$

Multiplication by $\frac{r^2}{R(r)\Theta(\theta)}$ and a little rearranging gives

$$\frac{r}{R(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left(rR'(r)\right) + \lambda r^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \mu,$$

which results in two additional SL ODE eigenvalue problems.

Altogether, we now have three ODEs:

- the time-dependent problem

$$T''(t) = -\lambda c^2 T(t)$$

- and from

$$\frac{r}{R(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left(rR'(r)\right) + \lambda r^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \mu$$

we get the two singular Sturm–Liouville problems

  - $$\Theta''(\theta) = -\mu\Theta(\theta)$$
    with periodic BCs $\quad \Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi)$
  - $$r\frac{\mathrm{d}}{\mathrm{d}r}\left(rR'(r)\right) + \left(\lambda r^2 - \mu\right)R(r) = 0$$
    with singularity BCs $\quad R(a) = 0, \quad |R(0)| < \infty$

The first problem

$$\Theta''(\theta) = -\mu\Theta(\theta)$$

with periodic BCs has eigenvalues and eigenfunctions

$$\mu_n = n^2, \qquad \Theta_n(\theta) = c_1 \cos n\theta + c_2 \sin n\theta, \qquad n = 0, 1, 2, \ldots$$

The second problem is more easily investigated if we first re-write it. Using the product rule we have

$$0 = r\frac{\mathrm{d}}{\mathrm{d}r}\left(rR'(r)\right) + \left(\lambda r^2 - \mu\right)R(r) = r^2R''(r) + rR'(r) + \left(\lambda r^2 - \mu\right)R(r).$$

One can use the Rayleigh quotient to show that λ must be positive, and so we can do a variable substitution $z = \sqrt{\lambda}r$.
Note that, by the chain rule, we then have

$$\frac{\mathrm{d}R}{\mathrm{d}r} = \frac{\mathrm{d}R}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{\mathrm{d}R}{\mathrm{d}z}\sqrt{\lambda}$$

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2} = \frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathrm{d}R}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{d}R}{\mathrm{d}z}\sqrt{\lambda}\right] = \frac{\mathrm{d}^2R}{\mathrm{d}z^2}\lambda$$

Therefore, if we apply the substitution $z = \sqrt{\lambda}r$ and the eigenvalues $\mu_n = n^2$ to the equation

$$r^2 R''(r) + rR'(r) + \left(\lambda r^2 - \mu_n\right) R(r) = 0, \qquad n = 0, 1, 2, \ldots$$

we get

$$\frac{z^2}{\lambda}\lambda R''(z) + \frac{z}{\sqrt{\lambda}}\sqrt{\lambda}R'(z) + \left(\lambda\frac{z^2}{\lambda} - n^2\right) R(z) = 0$$

$$\iff \quad z^2 R''(z) + zR'(z) + \left(z^2 - n^2\right) R(z) = 0, \quad n = 0, 1, 2, \ldots$$

This is known as Bessel's equation.

We will now solve Bessel's equation (you may have already seen this in MATH 252).

Solution of Bessel's equation

We assume the solution is given as a power series of the form

$$R(z) = z^c \sum_{j=0}^{\infty} a_j z^j = \sum_{j=0}^{\infty} a_j z^{j+c} \tag{7}$$

Assuming this series is differentiable, we compute the required derivatives

$$\begin{aligned} R'(z) &= \sum_{j=0}^{\infty} (j+c) a_j z^{j+c-1} \\ R''(z) &= \sum_{j=0}^{\infty} (j+c)(j+c-1) a_j z^{j+c-2} \end{aligned}$$

Inserting the power series *Ansatz* (7) and its derivatives into Bessel's equation

$$z^2 R''(z) + zR'(z) + \left(z^2 - n^2\right) R(z) = 0$$

we get

$$z^2 \sum_{j=0}^{\infty} (j+c)(j+c-1)a_j z^{j+c-2} + z \sum_{j=0}^{\infty} (j+c)a_j z^{j+c-1} + \left(z^2 - n^2\right) \sum_{j=0}^{\infty} a_j z^{j+c} = 0$$

or

$$\sum_{j=0}^{\infty} (j+c)(j+c-1)a_j z^{j+c} + \sum_{j=0}^{\infty} (j+c)a_j z^{j+c} + \left(z^2 - n^2\right) \sum_{j=0}^{\infty} a_j z^{j+c} = 0$$

$$\iff \quad \sum_{j=0}^{\infty} \left[(j+c)(j+c-1) + (j+c) - n^2\right] a_j z^{j+c} + \sum_{j=0}^{\infty} a_j z^{j+c+2} = 0$$

$$\iff \quad \sum_{j=0}^{\infty} \left[(j+c)^2 - n^2\right] a_j z^{j+c} + \sum_{j=2}^{\infty} a_{j-2} z^{j+c} = 0$$

Now we can divide out the factor z^c and get

$$\sum_{j=0}^{\infty} \left[(j+c)^2 - n^2 \right] a_j z^j + \sum_{j=2}^{\infty} a_{j-2} z^j = 0$$

or

$$\left(c^2 - n^2\right) a_0 + \left[(1+c)^2 - n^2\right] a_1 z + \sum_{j=2}^{\infty} \left\{ \left[(j+c)^2 - n^2\right] a_j + a_{j-2} \right\} z^j = 0.$$

In order to determine the unknown coefficients a_j in the power series of R we now compare coefficients of like powers of z.

From above

$$\left(c^2 - n^2\right) a_0 + \left[(1+c)^2 - n^2\right] a_1 z + \sum_{j=2}^{\infty} \left\{ \left[(j+c)^2 - n^2\right] a_j + a_{j-2} \right\} z^j = 0.$$

- Coefficient of z^0:

$$\left(c^2 - n^2\right) a_0 = 0$$
$$\Longrightarrow \quad a_0 = 0 \quad \text{or} \quad c = \pm n$$

  Since we don't want $a_0 = 0$ (see the explanation below) we have $c = \pm n$.

- Coefficient of z^1:

$$\left[(1+c)^2 - n^2\right] a_1 = 0$$
$$\Longrightarrow \quad \left[(1 \pm n)^2 - n^2\right] a_1 = 0$$
$$\Longrightarrow \quad (1 \pm 2n) a_1 = 0 \quad \Longrightarrow \quad a_1 = 0$$

  since we can't choose n (and n is a nonnegative integer).

For the following discussion we assume $c = +n$.

- Coefficient of z^j, $j > 1$:

$$\left[(j+n)^2 - n^2\right] a_j + a_{j-2} = 0$$

$$\iff \quad \left(j^2 + 2nj\right) a_j + a_{j-2} = 0$$

$$\iff \quad a_j = \frac{-1}{j^2 + 2nj} a_{j-2}, \quad j = 2, 3, 4 \ldots$$

This is a recurrence relation which requires two initial values: a_0 and a_1.

The recurrence relation

- couples all coefficients with even subscript (starting with a_0),
- and all those with odd subscripts (starting with a_1). Since $a_1 = 0$ we immediately know that

$$a_{2k+1} = 0, \qquad k = 1, 2, 3, \ldots$$

Now we can see why we didn't want to allow $a_0 = 0$ above. This would have resulted in a trivial solution $R(z) = 0$.

Let's calculate the coefficients a_j with even subscripts using the recurrence relation $a_j = \frac{-1}{j^2+2nj} a_{j-2}$, $j = 2, 3, 4, \ldots$.

$$a_2 = \frac{-1}{2^2 + 2n2} a_0 = \frac{-1}{4 + 4n} a_0 = \frac{-1}{1(n+1)2^2} a_0$$

$$a_4 = \frac{-1}{4^2 + 2n4} a_2 = \frac{-1}{16 + 8n} a_2 = \frac{-1}{2(n+2)2^2} a_2 = \frac{(-1)^2}{1 \cdot 2(n+1)(n+2)(2^2)^2} a_0$$

$$a_6 = \frac{-1}{6^2 + 2n6} a_4 = \frac{-1}{36 + 12n} a_4 = \frac{-1}{3(n+3)2^2} a_4 = \frac{(-1)^3}{1 \cdot 2 \cdot 3(n+1)(n+2)(n+3)(2^2)^3} a_0$$

$$a_{2k} = \frac{(-1)^k}{k!(n+1)(n+2)\cdots(n+k)(2^2)^k} a_0, \qquad k = 1, 2, 3, \ldots$$

Going back to the power series (7) for R we now know that

$$\begin{aligned} R(z) &= z^c \sum_{j=0}^{\infty} a_j z^j \\ &= z^n \sum_{k=0}^{\infty} a_{2k} z^{2k} = \sum_{k=0}^{\infty} a_{2k} z^{2k+n} \end{aligned}$$

with $a_{2k} = \frac{(-1)^k}{k!(n+1)(n+2)\cdots(n+k)(2^2)^k} a_0$, $k = 1, 2, 3, \ldots$

We now look at the radius of convergence of this power series using the ratio test:

$$\begin{aligned} &\lim_{k\to\infty} \left| \frac{a_{2(k+1)} z^{2(k+1)+n}}{a_{2k} z^{2k+n}} \right| \\ &= \lim_{k\to\infty} \frac{|z|^{2k+2+n}}{(k+1)!(n+1)(n+2)\cdots(n+k+1)2^{2k+2}} \frac{k!(n+1)(n+2)\cdots(n+k)2^{2k}}{|z|^{2k+n}} \\ &= \left|\frac{z}{2}\right|^2 \lim_{k\to\infty} \frac{1}{(k+1)(n+k+1)} = 0 \end{aligned}$$

Therefore the series converges for all z.

After all this work we are still free to choose the value of a_0.
Since

$$a_{2k} = \frac{(-1)^k}{k!(n+1)(n+2)\cdots(n+k)2^{2k}} a_0$$

the choice $a_0 = \frac{1}{n!2^n}$ gives us (using the convention that $0! = 1$)

$$a_{2k} = \frac{(-1)^k}{k!(n+1)(n+2)\cdots(n+k)2^{2k}} \frac{1}{n!2^n} = \frac{(-1)^k}{k!(n+k)!2^{2k+n}}$$

and therefore

$$\begin{aligned} R(z) &= \sum_{k=0}^{\infty} a_{2k} z^{2k+n} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!2^{2k+n}} z^{2k+n} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{z}{2}\right)^{2k+n} \end{aligned}$$

We now have found the Bessel functions of the first kind of order n:

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{z}{2}\right)^{2k+n}, \qquad n = 0, 1, 2, \ldots$$

Remark

Even though the Bessel functions J_n are defined only via a power series expansion, much is known about them.

- *In particular, as we just saw, they are the eigenfunctions of a (singular) Sturm–Liouville problem.*
- *They are one of the most popular so-called special functions, and much information is collected in, e.g., [Abramowitz & Stegun].*
- *Note that the functions we found here are J-Bessel functions. There are also Y-, I-, and K-Bessel functions.*
- *The Bessel functions we computed have positive integer order. There are also families of Bessel functions with negative integer order, or even real or complex order.*
- *Software packages such as* MATLAB*, MuPAD, Maple or Mathematica all have special routines for Bessel functions.*

In addition to being able to evaluate the Bessel functions J_n, we will need to know their zeros.
It is known that each Bessel function J_n, $n = 0, 1, 2, \ldots$, has infinitely many distinct zeros that can be ordered $z_{n,1} < z_{n,2} < \ldots$. They are not equally spaced.

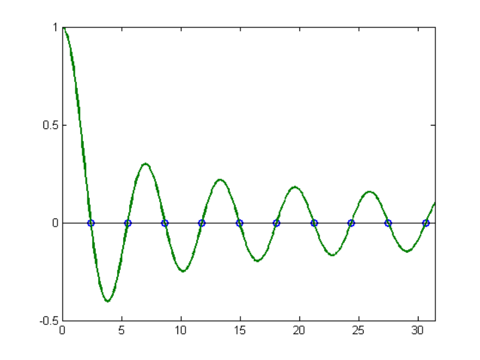


Figure: The Bessel function J_0.

Returning to our 3 ODEs...

- Angular eigenvalue problem: Earlier, we already decided that

$$\Theta''(\theta) = -\mu\Theta(\theta)$$
$$\Theta(-\pi) = \Theta(\pi), \qquad \Theta'(-\pi) = \Theta'(\pi)$$

has eigenvalues and eigenfunctions

$$\mu_n = n^2 \quad \text{and} \quad \Theta_n(\theta) = c_1 \cos n\theta + c_2 \sin n\theta, \qquad n = 0, 1, 2, \ldots$$

- Radial eigenvalue problem: Moreover, we've now found that the eigenfunctions of

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(rR'(r)\right) + \left(\lambda r^2 - n^2\right) R(r) = 0$$
$$R(a) = 0, \qquad |R(0)| < \infty$$

are (since we substituted $z = \sqrt{\lambda}r$ in Bessel's equation)

$$R_n(z) = J_n(\sqrt{\lambda}r) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{\sqrt{\lambda}r}{2}\right)^{2k+n}, \qquad n = 0, 1, 2, \ldots$$

- Radial eigenvalue problem (cont.): Now the BCs tell us that the eigenvalues $\lambda_{n,m}$ are such that

$$R_n(a) = J_n(\sqrt{\lambda_{n,m}}a) = 0,$$

i.e., $\sqrt{\lambda_{n,m}}a$ is the m-th zero of the Bessel function J_n, or

$$\lambda_{n,m} = \left(\frac{z_{n,m}}{a}\right)^2, \qquad n = 0, 1, 2, \ldots, \; m = 1, 2, 3, \ldots$$

where $z_{n,m}$ is the m-th zero of the Bessel function of order n, i.e.,

$$J_n(z_{n,m}) = 0.$$

- Time equation: We also know that since $\lambda_{n,m} > 0$

$$T''(t) = -\lambda_{n,m} c^2 T(t)$$

has general solution

$$T_{n,m}(t) = c_1 \cos\left(\sqrt{\lambda_{n,m}} ct\right) + c_2 \sin\left(\sqrt{\lambda_{n,m}} ct\right).$$

Therefore, superposition requires the solution to be of the form

$$\begin{aligned} u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Big[& a_{n,m}J_n(\sqrt{\lambda_{n,m}}r)\cos n\theta \cos\left(\sqrt{\lambda_{n,m}}ct\right) \\ & + b_{n,m}J_n(\sqrt{\lambda_{n,m}}r)\cos n\theta \sin\left(\sqrt{\lambda_{n,m}}ct\right) \\ & + c_{n,m}J_n(\sqrt{\lambda_{n,m}}r)\sin n\theta \cos\left(\sqrt{\lambda_{n,m}}ct\right) \\ & + d_{n,m}J_n(\sqrt{\lambda_{n,m}}r)\sin n\theta \sin\left(\sqrt{\lambda_{n,m}}ct\right)\Big] \end{aligned}$$

and the (Fourier) coefficients can be found using the initial conditions.

We now illustrate this with an example.

Example (Vibration of a circularly symmetric drum with zero initial velocity)

Because of circular symmetry there is no change in the angular variable and the wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right), \qquad 0 < r < a, \ t > 0$$

with boundary conditions

$$u(a, t) = 0 \quad \text{and} \quad |u(0, t)| < \infty$$

and initial conditions

$$u(r, 0) = f(r) \quad \text{and} \quad \frac{\partial u}{\partial t}(r, 0) = 0.$$

Example ((cont.))

For separation of variables we require only a two-way split, so

$$u(r,t) = R(r)T(t),$$

and our resulting ODEs are

$$T''(t) = -\lambda c^2 T(t)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(rR'(r)\right) + \lambda r R(r) = 0$$
$$R(a) = 0 \quad \text{and} \quad |R(0)| < \infty.$$

Example ((cont.))

Using the product rule we can rewrite the radial ODE

$$\frac{d}{dr}\left(rR'(r)\right) + \lambda r R(r) = 0$$

as

$$rR''(r) + R'(r) + \lambda r R(r) = 0$$

and then multiply by r and do the substitution $z = \sqrt{\lambda} r$ as before to recognize

$$r^2 R''(r) + rR'(r) + \lambda r^2 R(r) = 0$$

$$\stackrel{z=\sqrt{\lambda}r}{\Longrightarrow} \qquad z^2 R''(z) + zR'(z) + z^2 R(z) = 0$$

as Bessel's equation for the case $n = 0$, i.e., for J_0.

Example ((cont.))

Therefore we have the solution

$$R(z) = J_0(\sqrt{\lambda_n} r)$$

with $\sqrt{\lambda_n} a$ the n-th zero of the Bessel function J_0 (all of which are positive).
Inserting these eigenvalues into the time-equation we get the solutions

$$T_n(t) = c_1 \cos \sqrt{\lambda_n} ct + c_2 \sin \sqrt{\lambda_n} ct$$

and superposition gives us

$$u(r,t) = \sum_{n=1}^{\infty} \left[a_n \cos \sqrt{\lambda_n} ct + b_n \sin \sqrt{\lambda_n} ct \right] J_0(\sqrt{\lambda_n} r).$$

Example ((cont.))

The first initial condition gives

$$u(r,0) = \sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_n} r) \stackrel{!}{=} f(r).$$

This is a Fourier-Bessel series with coefficients

$$a_n = \frac{\int_0^a f(r) J_0(\sqrt{\lambda_n} r) r \mathrm{d}r}{\int_0^a J_0^2(\sqrt{\lambda_n} r) r \mathrm{d}r}.$$

Note the role of the weight $\sigma(r) = r$ from the SL equation in the integrals.

Example ((cont.))

Similarly,

$$\frac{\partial u}{\partial t}(r,0) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} c b_n J_0(\sqrt{\lambda_n} r) \stackrel{!}{=} 0$$

with

$$b_n = \frac{1}{c\sqrt{\lambda_n}} \frac{\int_0^a 0 J_0(\sqrt{\lambda_n} r) r dr}{\int_0^a J_0^2(\sqrt{\lambda_n} r) r dr} = 0.$$

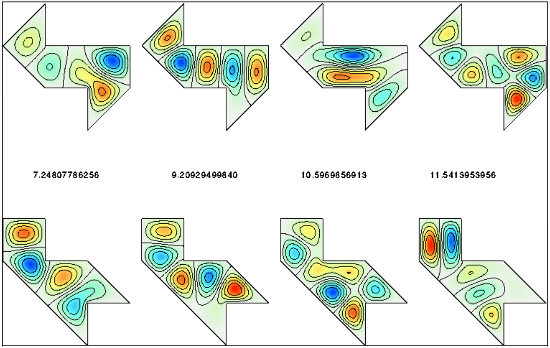
Remark

This problem is illustrated in the Mathematica notebook `Drum.nb`*. The notebook also contains an illustration of the modes and a second example (vibration of a rectangular drum).*

Isospectral Drums

In the 1960s Mark Kac (at the time a mathematician at Rockefeller University in New York) asked the question "Can one hear the shape of a drum?" [Kac (1966)].

The answer to this inverse problem was not provided until the 1990s by Carolyn Gordon, David Webb and Scott Wolpert in a paper entitled "One Cannot Hear the Shape of a Drum" [GWW (1992)].



Detailed numerical computations illustrating this problem were presented in [Driscoll (1997)] (see also [Peterson (1997)]).

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