MATH 461: Fourier Series and Boundary Value Problems

Chapter IV: Vibrating Strings and Membranes

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Outline

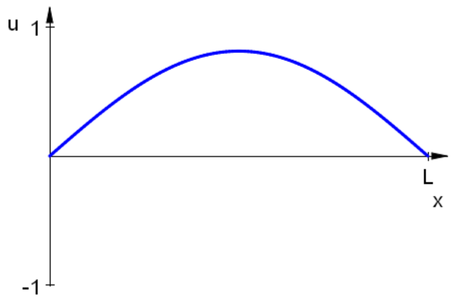
Vibrating Strings

We will now derive a new mathematical model.

Consider a stretched elastic string of length L with equilibrium position along the x-axis.
Every point $(x, 0)$, $0 \leq x \leq L$, of the string has a displacement

$$y = u(x, t)$$

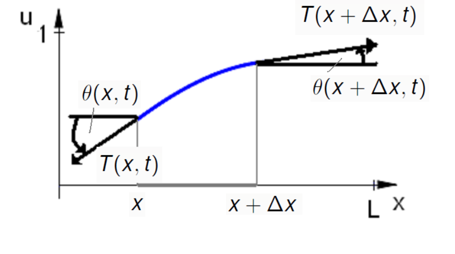
at any given time $t \geq 0$.



Slow Normal Fast Play/Pause Stop

In order to be able to come up with a reasonable (and manageable) mathematical model we need to make some simplifying assumptions:

- We consider only small displacements (relative to the length of the string).
- This implies that we can neglect horizontal displacements.
- We assume a perfectly flexible string, i.e., only tangential forces are acting on the string (see the figure).



Here we denote by

- T the (tangential) tension,
- θ the angle T forms with the horizontal (measured counter-clockwise).

For a small segment from x to $x + \Delta x$ we will use Newton's law

$$F = m\,a$$

coupled with a conservation law that balances the forces in a segment of the string as

$$\{\text{total force according to Newton}\}$$
$$=$$
$$\{\text{vertical force at left end of segment}\}$$
$$+$$
$$\{\text{vertical force at right end of segment}\}$$
$$+$$
$$\{\text{vertical component of body force (or possible external force)}\}$$

or

$$m\,a = V_L + V_R + V_{body}$$

to derive a PDE for the displacement $u = u(x,t)$.

To this end we need

- mass:

$$m(x) = \underbrace{\rho_0(x)}_{\text{density}} \Delta x$$

- acceleration:

$$a(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t)$$

- body force:

$$V_{body}(x,t) = \rho_0(x)\Delta x\, Q(x,t)$$

  with $Q(x,t)$ the vertical component of the body force per unit mass

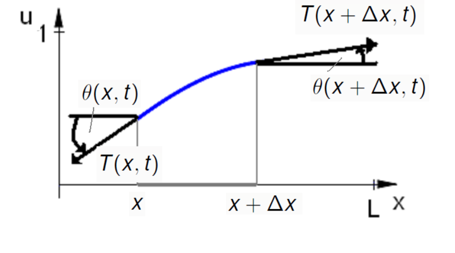
A commonly used body force is $V_{body} = -m\,g$.

We also need to carefully study the tensile forces in the string.

Since the displacement and motion are assumed to happen only in the vertical direction we require only the vertical component of the tensile force:

- At the left end:

$$V_L(x,t) = -T(x,t)\sin\theta(x,t)$$



- At the right end:

$$V_R(x,t) = T(x+\Delta x,t)\sin\theta(x+\Delta x,t)$$

Putting all of this together, the balance of force equation gives us

$$\begin{aligned}\rho_0(x)\Delta x \frac{\partial^2 u}{\partial t^2}(x,t) &= V_L(x,t) + V_R(x,t) + \rho_0(x)\Delta x\, Q(x,t) \\ &= -T(x,t)\sin\theta(x,t) + T(x+\Delta x,t)\sin\theta(x+\Delta x,t) \\ &\quad +\rho_0(x)\Delta x\, Q(x,t)\end{aligned}$$

If we divide by Δx and let Δx go to zero we get

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2}(x,t) = \underbrace{\frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)}{\Delta x}}_{\to \frac{\partial}{\partial x}[T(x,t)\sin\theta(x,t)] \text{ as } \Delta x \to 0} + \rho_0(x)Q(x,t)$$

i.e.,

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial}{\partial x}\left[T(x,t)\sin\theta(x,t)\right] + \rho_0(x)Q(x,t).$$

Since the vertical displacement is assumed to be small, the angle $\theta(x, t)$ is also small (so that $\cos\theta(x, t) \approx 1$).
Therefore,

$$\sin\theta(x, t) \approx \frac{\sin\theta(x, t)}{\cos\theta(x, t)} = \tan\theta(x, t)$$

Here it is important to note that $\tan\theta(x, t)$ corresponds to the "slope" of the string, i.e.,

$$\tan\theta(x, t) = \frac{\partial u}{\partial x}(x, t).$$

Thus

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2}(x, t) = \frac{\partial}{\partial x}\left[T(x, t)\sin\theta(x, t)\right] + \rho_0(x)Q(x, t)$$

turns into the PDE for the vibrating string

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2}(x, t) = \frac{\partial}{\partial x}\left[T(x, t)\frac{\partial u}{\partial x}(x, t)\right] + \rho_0(x)Q(x, t).$$

In order to get the commonly used version of the 1D wave equation we make two further assumptions:

- Small displacements and perfectly elastic strings imply an approximately constant tension, i.e.,

$$T(x,t) \approx T_0 = \text{const.}$$

  So

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2}(x,t) = T_0 \frac{\partial^2 u}{\partial x^2}(x,t) + \rho_0(x)Q(x,t)$$

- Often the external force term can be neglected because the weight of the string is small compared to the tension so that there is no "sag" in the string. Then we have

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \quad \text{with } c^2 = \frac{T_0}{\rho_0(x)},$$

  the 1D wave equation.

Since the 1D wave equation contains

- a second-order spatial derivative term and
- a second-order time derivative term

we need to specify

- two spatial conditions (as usual, we do this in the form of boundary conditions – see next section), and
- two temporal conditions. These will be given as
  - initial position: $u(x,0) = f(x)$ and
  - initial velocity: $\frac{\partial u}{\partial t}(x,0) = g(x)$.

Fixed Ends

The simplest form of boundary conditions is

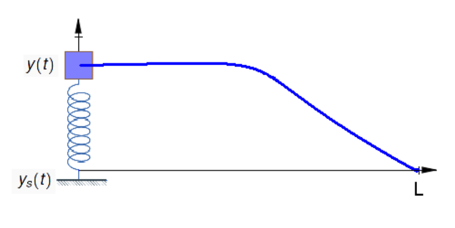
$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

We could also have ends that are affixed to a moving support. Then they would be of the form

$$\begin{aligned} u(0,t) &= f_1(t) \\ u(L,t) &= f_2(t) \end{aligned}$$

Elastic Ends

This will occur if a string is, e.g., attached to a spring-mass system



Thus

$$u(0,t) = y(t),$$

where $y(t)$ is unknown. In fact, y is determined by an ODE for a spring-mass system with a possibly moving support $y_s(t)$.
Note that – to keep things manageable – we assume that the spring-mass system moves only vertically.

The ODE for y is obtained by combining Newton's and Hooke's laws:

$$m\frac{d^2y}{dt^2}(t) = -k\left[y(t) - y_s(t) - \ell\right] + \text{other forces},$$

where

- k is the spring constant,
- $y_s(t)$ denotes the moving support of the spring, possibly driven by some external force,
- ℓ is the length of the unstretched spring.

The "other forces" should include the vertical component of the tensile force of the string:

$$T(0,t)\sin\theta(0,t) \overset{\text{small }\theta}{\approx} T_0\tan\theta(0,t) = T_0\frac{\partial u}{\partial x}(0,t).$$

Thus, elastic BCs are of the form

$$m\frac{d^2y}{dt^2}(t) = -k\left[y(t) - y_s(t) - \ell\right] + T_0\frac{\partial u}{\partial x}(0,t) + g(t).$$

In the special case

- $g(t) = 0$ (i.e., no additional external forces)
- with a spring-mass system with small mass, i.e., $m \approx 0$,
- and taking $y(t) = u(0, t)$

we get the BC

$$\begin{aligned} k\left[u(0,t) - y_s(t) - \ell\right] &= T_0 \frac{\partial u}{\partial x}(0,t) \\ \iff \quad k\left[u(0,t) - y_E(t)\right] &= T_0 \frac{\partial u}{\partial x}(0,t), \end{aligned}$$

where $y_E(t) = y_s(t) + \ell$ denotes the equilibrium position of the spring-mass system.

If in addition $y_E(t) = 0$, i.e., the equilibrium position of the spring-mass system coincides with the equilibrium position of the string and occurs at the origin, then

$$k\left[u(0,t) - y_E(t)\right] = T_0 \frac{\partial u}{\partial x}(0,t)$$

becomes

$$T_0 \frac{\partial u}{\partial x}(0,t) = k u(0,t).$$

Remark

Note that this BC is analogous to the one we obtained via Newton's law of cooling for the 1D heat equation.

Remark

Note that the tensile force and spring constant are usually positive, i.e., $T_0 > 0$, $k > 0$. Then we see from

$$T_0 \frac{\partial u}{\partial x}(0,t) = ku(0,t) \qquad \text{that} \qquad u(0,t) > 0 \Longleftrightarrow \frac{\partial u}{\partial x}(0,t) > 0.$$

However, at the right end, $x = L$, we will have

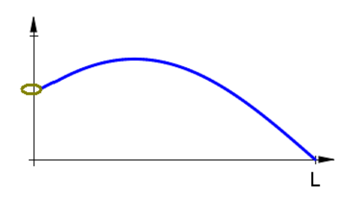
$$u(L,t) > 0 \Longleftrightarrow \frac{\partial u}{\partial x}(L,t) < 0,$$

i.e., upward motion pulls the string "inside". Therefore, the corresponding BC at the right end has a different sign and looks like

$$T_0 \frac{\partial u}{\partial x}(L,t) = -ku(L,t).$$

Free Ends

This will happen when the string is attached to a ring that slides frictionless along a vertical support.



Physically, this corresponds to the limiting case of the spring-mass system for $k \to 0$, i.e.,

$$T_0 \frac{\partial u}{\partial x}(0, t) = 0.$$

Remark

Note the analogy to the insulated end condition for the heat equation.

We solve the PDE

$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2\frac{\partial^2}{\partial x^2}u(x,t), \qquad \text{for } 0 < x < L,\ t > 0, \tag{1}$$

with boundary conditions

$$u(0,t) = u(L,t) = 0 \qquad \text{for } t > 0 \tag{2}$$

and initial conditions

$$u(x,0) = f(x) \qquad \text{for } 0 < x < L \quad \text{(initial position)} \tag{3}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \qquad \text{for } 0 < x < L \quad \text{(initial velocity)} \tag{4}$$

From our earlier derivations the constant $c^2 = \frac{T_0}{\rho_0}$ is given as the ratio of tension to density.

Remark

Since the PDE and its BCs are linear and homogeneous we attempt to solve the problem using separation of variables.

The usual *Ansatz* $u(x,t) = \varphi(x)T(t)$ gives us the partial derivatives

$$\begin{aligned}\frac{\partial^2}{\partial t^2}u(x,t) &= \varphi(x)T''(t)\\ \frac{\partial^2}{\partial x^2}u(x,t) &= \varphi''(x)T(t)\end{aligned}$$

so that we have

$$\varphi(x)T''(t) = c^2\varphi''(x)T(t)$$

or

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda.$$

The resulting two ODEs are

$$T''(t) = -\lambda c^2 T(t) \tag{5}$$

and

$$\varphi''(x) = -\lambda \varphi(x) \tag{6}$$

with BCs

$$\varphi(0) = \varphi(L) = 0. \tag{7}$$

Remark

We chose $-\lambda$ for the separation constant above so that (6)-(7) is one of our standard boundary-value problems with well-known eigenvalues and eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad \varphi_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

Remark

Moreover, the ODE (5) for the time-dependent component T has the physically meaningful, oscillating solution for positive λ, i.e.,

$$T(t) = c_1 \cos c\sqrt{\lambda}t + c_2 \sin c\sqrt{\lambda}t.$$

For this example the other two types of solution

- $T(t) = c_1 t + c_2 \qquad \text{for } \lambda = 0$
- $T(t) = c_1 e^{c\sqrt{-\lambda}t} + c_2 e^{-c\sqrt{-\lambda}t} \qquad \text{for } \lambda < 0$

are not relevant.

Using the principle of superposition to combine the solutions of the ODEs (5) and (6) we get

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{cn\pi t}{L} + B_n \sin \frac{cn\pi t}{L} \right] \sin \frac{n\pi x}{L}$$

To determine the expansion coefficients A_n and B_n we use the initial conditions (3) and (4).
Since $\cos 0 = 1$ and $\sin 0 = 0$ we have

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \stackrel{!}{=} f(x)$$

so that

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, \mathrm{d}x.$$

In order to apply the initial condition (4) we need to ensure that

- u is continuous and
- $\frac{\partial u}{\partial t}$ is piecewise smooth.

Then we can apply term-by-term differentiation to get

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty}\left[-A_n\frac{cn\pi}{L}\sin\frac{cn\pi t}{L} + B_n\frac{cn\pi}{L}\cos\frac{cn\pi t}{L}\right]\sin\frac{n\pi x}{L}$$

and

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n\frac{cn\pi}{L}\sin\frac{n\pi x}{L} \overset{!}{=} g(x)$$

so that

$$B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x.$$

The solution for this problem is illustrated in the Mathematica notebook `Wave.nb`.

Normal Modes and Overtones: Applications to Music

For each n, the n-th term of the Fourier series solution

$$u_n(x,t) = \left[A_n \cos\frac{cn\pi t}{L} + B_n \sin\frac{cn\pi t}{L}\right] \sin\frac{n\pi x}{L}$$

is called the n-th normal mode of the solution.

The term u_n describes a harmonic motion with frequency

$$f_n = \frac{cn\pi}{L}/2\pi = \frac{cn}{2L}$$

or with circular frequency $\omega = \frac{cn\pi}{L}$.

The frequencies f_n are called the natural frequencies of the solution u.

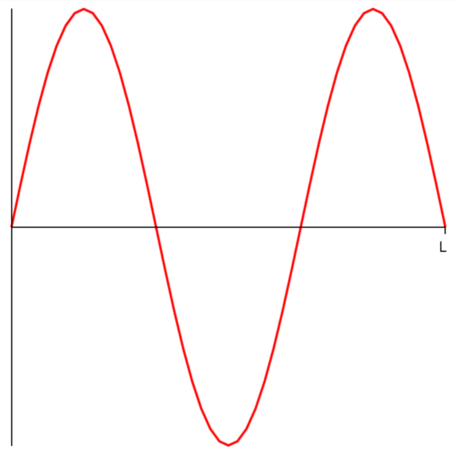


Figure: Third mode, or second overtone, $n = 3$.

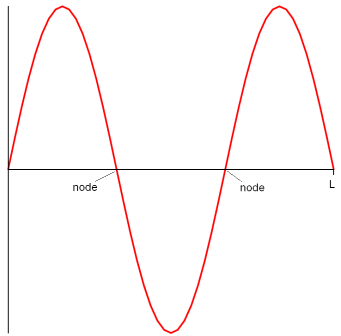
Note that each one of the modes will be zero for all t if

$$\sin \frac{n\pi x}{L} = 0, \quad \text{i.e.,} \ \frac{n\pi x}{L} = k\pi, \quad k = 1, 2, 3, \ldots$$

The points

$$x_k = k\frac{L}{n}, \quad k = 1, 2, \ldots, n$$

are called nodes.



By placing their finger at a node point, players of string instruments such as guitars, violins, etc., can produce so-called flageolet tones.

A string instrument can be tuned, i.e., the pitch can be made higher by

- decreasing the length of the string, L, or
- increasing c. Since $c = \sqrt{\frac{T_0}{\rho_0}}$ this means that one can either
  - increase the tension or
  - decrease the density of the string.

Remark

Overtones are illustrated acoustically in the MATLAB *script* `overtones.m`.

- *We can hear there that the first overtone (i.e., with double the frequency) is an octave higher than the fundamental mode.*
- *Similarly, the second overtone is a fifth higher than the first overtone,*
- *and the third overtone is a fourth higher than the second overtone.*

Traveling Waves (Exercise 4.4.7)

Let's again consider the 1D wave equation

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}u(x,t) &= c^2\frac{\partial^2}{\partial x^2}u(x,t)\\
u(0,t) &= u(L,t) = 0\\
u(x,0) &= f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x) = 0
\end{aligned}$$

From the general solution derived above we know that

$$B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx = 0.$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{cn\pi t}{L}\sin\frac{n\pi x}{L}$$

with

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

The trigonometric identity

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

with $\alpha = \frac{n\pi x}{L}$ and $\beta = \frac{cn\pi t}{L}$ gives us

$$\begin{aligned}\cos\frac{cn\pi t}{L}\sin\frac{n\pi x}{L} &= \frac{1}{2}\left[\sin\left(\frac{n\pi x}{L} + \frac{cn\pi t}{L}\right) + \sin\left(\frac{n\pi x}{L} - \frac{cn\pi t}{L}\right)\right] \\ &= \frac{1}{2}\left[\sin\frac{n\pi}{L}(x+ct) + \sin\frac{n\pi}{L}(x-ct)\right]\end{aligned}$$

So the solution

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{cn\pi t}{L}\sin\frac{n\pi x}{L}$$

becomes

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}(x+ct) + \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}(x-ct)$$

Remark

Note that the two parts of

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}(x+ct) + \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}(x-ct)$$

are Fourier sine series of f evaluated at $x+ct$ and $x-ct$, respectively.

Therefore

$$u(x,t) = \frac{1}{2}\left[\bar{f}(x+ct) + \bar{f}(x-ct)\right],$$

where $\bar{f}$ is the odd $2L$-periodic extension of f.

Remark

This shows that we can get the solution of the 1D wave equation without actually summing the infinite series!

The solution of the 1D wave equation in the form

$$u(x,t) = \frac{1}{2}\left[\bar{f}(x+ct) + \bar{f}(x-ct)\right],$$

is known as d'Alembert's solution, after Jean d'Alembert who first formulated the 1D wave equation and proposed this form of the solution in 1746 – 22 years before Fourier's birth.

It can be interpreted as the average of two traveling waves:

- one traveling to the left,
- the other to the right

– both with speed c.

This is illustrated in the Mathematica notebook `Wave.nb`.

Remark

The traveling wave solution is closely related to the solution of PDEs by the method of characteristics (see Chapter 12 in [Haberman] and MATH 489).

References I

R. Haberman.
Applied Partial Differential Equations.
Pearson (5th ed.), Upper Saddle River, NJ, 2012.