MATH 461: Fourier Series and Boundary Value Problems

Chapter IV: Vibrating Strings and Membranes

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Outline

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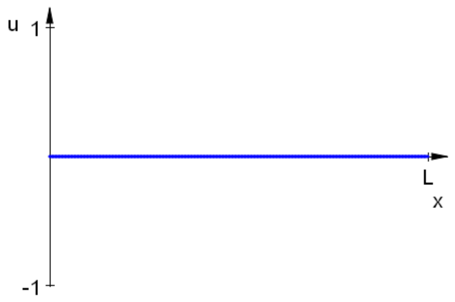

Vibrating Strings

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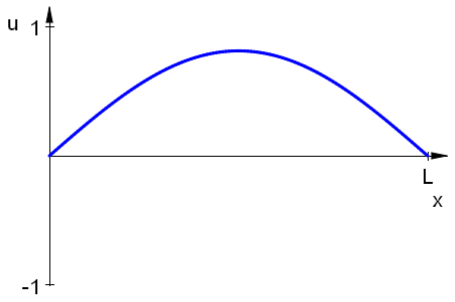
Consider a stretched elastic string of length L with equilibrium position along the x-axis.
Every point $(x, 0)$, $0 \leq x \leq L$, of the string has a displacement

$$y = u(x,t)$$

at any given time $t \geq 0$.



Slow Normal Fast Play/Pause Stop

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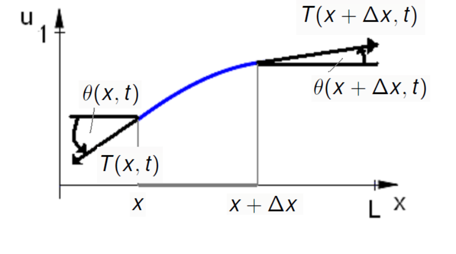

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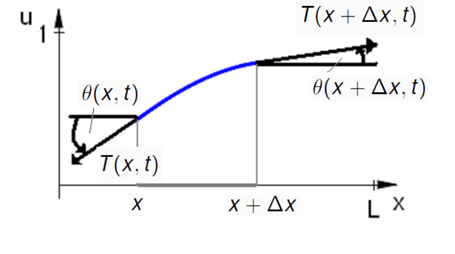

Here we denote by

- T the (tangential) tension,
- θ the angle T forms with the horizontal (measured counter-clockwise).

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coupled with a conservation law that balances the forces in a segment of the string as

$$\{\text{total force according to Newton}\}$$
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or

$$m\,a = V_L + V_R + V_{body}$$

to derive a PDE for the displacement $u = u(x, t)$.

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A commonly used body force is $V_{body} = -m\,g$.

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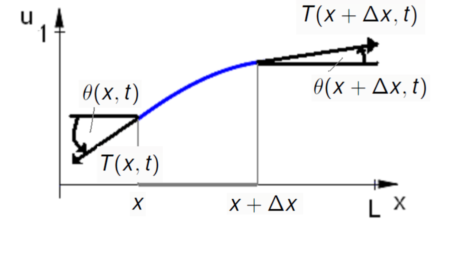

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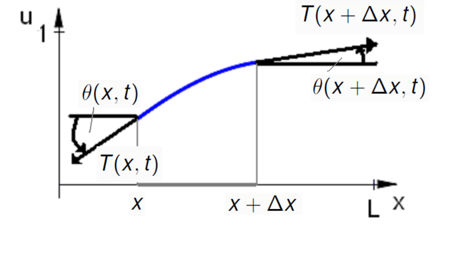

- At the left end:

$$V_L(x,t) = -T(x,t)\sin\theta(x,t)$$

- At the right end:

$$V_R(x,t) = T(x+\Delta x,t)\sin\theta(x+\Delta x,t)$$

Putting all of this together, the balance of force equation gives us

$$\rho_0(x)\Delta x \frac{\partial^2 u}{\partial t^2}(x,t) = V_L(x,t) + V_R(x,t) + \rho_0(x)\Delta x\, Q(x,t)$$

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If we divide by Δx we get

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\rho_0(x)\frac{\partial^2 u}{\partial t^2}(x,t) &= \frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)}{\Delta x} \\
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i.e.,

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial}{\partial x}\left[T(x,t)\sin\theta(x,t)\right] + \rho_0(x)Q(x,t).$$

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turns into the PDE for the vibrating string

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$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2\frac{\partial^2 u}{\partial x^2}(x,t) \quad \text{with } c^2 = \frac{T_0}{\rho_0(x)},$$

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- two temporal conditions. These will be given as
  - initial position: $u(x, 0) = f(x)$ and
  - initial velocity: $\frac{\partial u}{\partial t}(x, 0) = g(x)$.

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Fixed Ends

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We could also have ends that are affixed to a moving support. Then they would be of the form

$$\begin{aligned} u(0,t) &= f_1(t) \\ u(L,t) &= f_2(t) \end{aligned}$$

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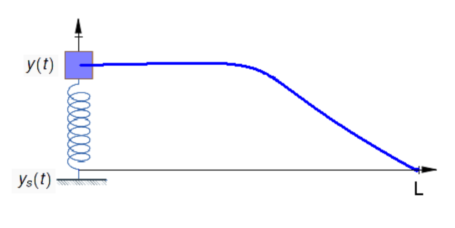

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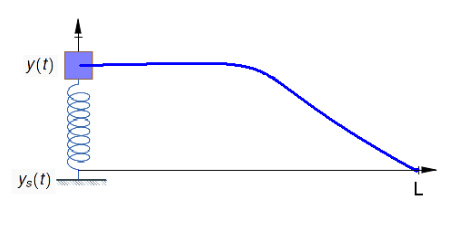

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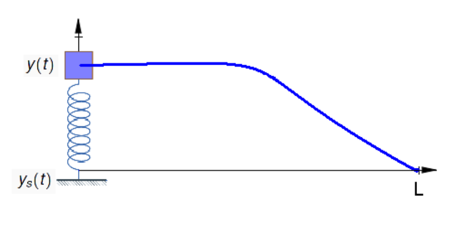

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where $y(t)$ is unknown. In fact, y is determined by an ODE for a spring-mass system with a possibly moving support $y_s(t)$.
Note that – to keep things manageable – we assume that the spring-mass system moves only vertically.

The ODE for y is obtained by combining Newton's and Hooke's laws:

$$m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}(t) = -k\left[y(t) - y_s(t) - \ell\right] + \text{other forces},$$

where

- k is the spring constant,
- $y_s(t)$ denotes the moving support of the spring, possibly driven by some external force,
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Thus, elastic BCs are of the form

$$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2}(t) = -k\left[y(t) - y_s(t) - \ell\right] + T_0\frac{\partial u}{\partial x}(0,t) + g(t).$$

In the special case

- $g(t) = 0$ (i.e., no additional external forces)
- with a spring-mass system with small mass, i.e., $m \approx 0$,
- and taking $y(t) = u(0, t)$

we get the BC

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$$\begin{aligned} k\left[u(0,t) - y_s(t) - \ell\right] &= T_0 \frac{\partial u}{\partial x}(0,t) \\ \iff \quad k\left[u(0,t) - y_E(t)\right] &= T_0 \frac{\partial u}{\partial x}(0,t), \end{aligned}$$

where $y_E(t) = y_s(t) + \ell$ denotes the equilibrium position of the spring-mass system.

If in addition $y_E(t) = 0$, i.e., the equilibrium position of the spring-mass system coincides with the equilibrium position of the string and occurs at the origin, then

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Remark

Note that this BC is analogous to the one we obtained via Newton's law of cooling for the 1D heat equation.

Remark

Note that the tensile force and spring constant are usually positive, i.e., $T_0 > 0$, $k > 0$. Then we see from

$$T_0 \frac{\partial u}{\partial x}(0,t) = ku(0,t) \qquad \text{that} \qquad u(0,t) > 0 \iff \frac{\partial u}{\partial x}(0,t) > 0.$$

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However, at the right end, $x = L$, *we will have*

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i.e., upward motion pulls the string "inside".

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However, at the right end, $x = L$, we will have

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i.e., upward motion pulls the string "inside". Therefore, the corresponding BC at the right end has a different sign and looks like

$$T_0 \frac{\partial u}{\partial x}(L,t) = -ku(L,t).$$

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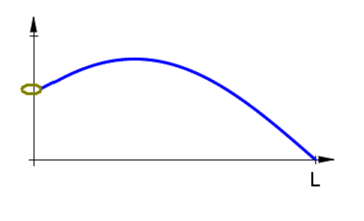

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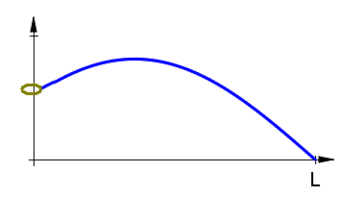

Physically, this corresponds to the limiting case of the spring-mass system for $k \to 0$, i.e.,

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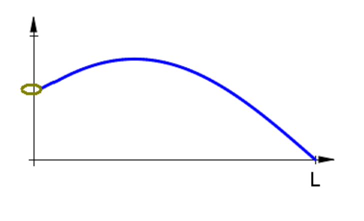

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Remark

Note the analogy to the insulated end condition for the heat equation.

Outline

We solve the PDE

$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2 \frac{\partial^2}{\partial x^2}u(x,t), \qquad \text{for } 0 < x < L,\ t > 0, \tag{1}$$

with boundary conditions

$$u(0,t) = u(L,t) = 0 \qquad \text{for } t > 0 \tag{2}$$

and initial conditions

$$u(x,0) = f(x) \qquad \text{for } 0 < x < L \quad \text{(initial position)} \tag{3}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \qquad \text{for } 0 < x < L \quad \text{(initial velocity)} \tag{4}$$

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From our earlier derivations the constant $c^2 = \frac{T_0}{\rho_0}$ is given as the ratio of tension to density.

Remark

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The usual *Ansatz* $u(x,t) = \varphi(x)T(t)$ gives us the partial derivatives

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so that we have

$$\varphi(x)T''(t) = c^2\varphi''(x)T(t)$$

or

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda.$$

The resulting two ODEs are

$$T''(t) = -\lambda c^2 T(t) \tag{5}$$

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We chose $-\lambda$ for the separation constant above so that (6)-(7) is one of our standard boundary-value problems with well-known eigenvalues and eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad \varphi_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

Remark

Moreover, the ODE (5) for the time-dependent component T has the physically meaningful, oscillating solution for positive λ, i.e.,

$$T(t) = c_1 \cos c\sqrt{\lambda}t + c_2 \sin c\sqrt{\lambda}t.$$

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For this example the other two types of solution

- $T(t) = c_1 t + c_2$ for $\lambda = 0$
- $T(t) = c_1 e^{c\sqrt{-\lambda}t} + c_2 e^{-c\sqrt{-\lambda}t}$ for $\lambda < 0$

are not relevant.

Using the principle of superposition to combine the solutions of the ODEs (5) and (6) we get

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{cn\pi t}{L} + B_n \sin \frac{cn\pi t}{L} \right] \sin \frac{n\pi x}{L}$$

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Since $\cos 0 = 1$ and $\sin 0 = 0$ we have

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$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

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The solution for this problem is illustrated in the Mathematica notebook `Wave.nb`.

Normal Modes and Overtones: Applications to Music

For each n, the n-th term of the Fourier series solution

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The frequencies f_n are called the natural frequencies of the solution u.

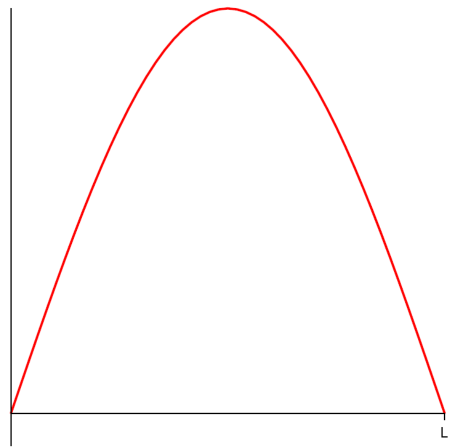

Figure: Fundamental mode, $n = 1$.

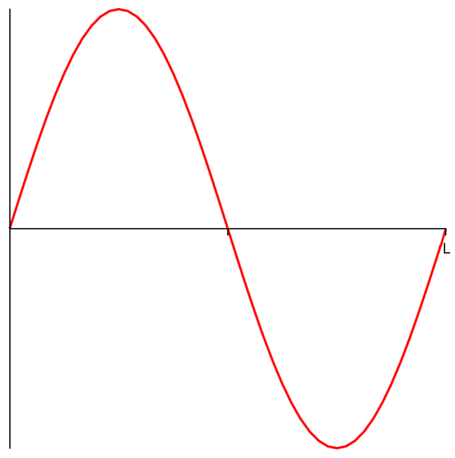

Figure: Second mode, or first overtone, $n = 2$.

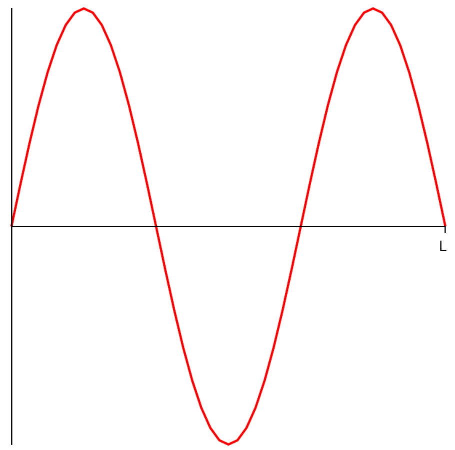

Figure: Third mode, or second overtone, $n = 3$.

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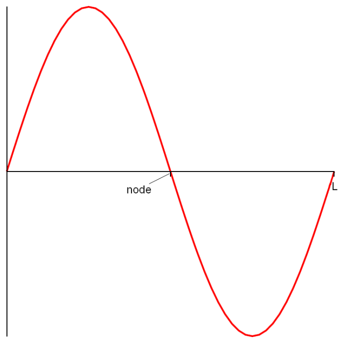
The points

$$x_k = k\frac{L}{n}, \quad k = 1, 2, \ldots, n$$

are called nodes.



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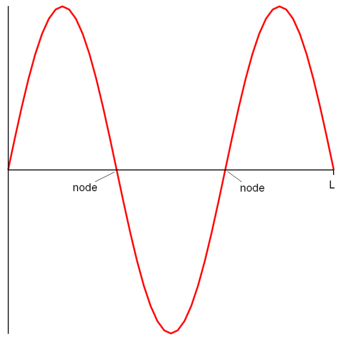

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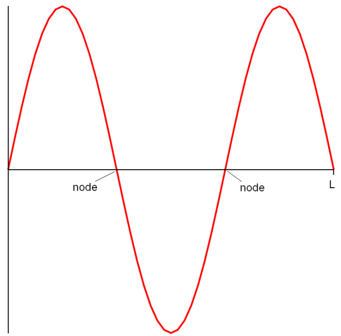

By placing their finger at a node point, players of string instruments such as guitars, violins, etc., can produce so-called flageolet tones.

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- *We can hear there that the first overtone (i.e., with double the frequency) is an octave higher than the fundamental mode.*
- *Similarly, the second overtone is a fifth higher than the first overtone,*
- *and the third overtone is a fourth higher than the second overtone.*

Traveling Waves (Exercise 4.4.7)

Let's again consider the 1D wave equation

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} u(x,t) &= c^2 \frac{\partial^2}{\partial x^2} u(x,t) \\
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From the general solution derived above we know that

$$B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx = 0.$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{cn\pi t}{L}\sin\frac{n\pi x}{L}$$

with

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The trigonometric identity

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

with $\alpha = \frac{n\pi x}{L}$ and $\beta = \frac{cn\pi t}{L}$ gives us

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Remark

This shows that we can get the solution of the 1D wave equation without actually summing the infinite series!

The solution of the 1D wave equation in the form

$$u(x,t) = \frac{1}{2}\left[\bar{f}(x+ct) + \bar{f}(x-ct)\right],$$

is known as d'Alembert's solution, after Jean d'Alembert who first formulated the 1D wave equation and proposed this form of the solution in 1746 – 22 years before Fourier's birth.

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The solution of the 1D wave equation in the form

$$u(x,t) = \frac{1}{2}\left[\bar{f}(x+ct) + \bar{f}(x-ct)\right],$$

is known as d'Alembert's solution, after Jean d'Alembert who first formulated the 1D wave equation and proposed this form of the solution in 1746 – 22 years before Fourier's birth.

It can be interpreted as the average of two traveling waves:

- one traveling to the left,
- the other to the right

– both with speed c.

This is illustrated in the Mathematica notebook `Wave.nb`.

Remark

The traveling wave solution is closely related to the solution of PDEs by the method of characteristics (see Chapter 12 in [Haberman] and MATH 489).

Outline

References I

R. Haberman.
Applied Partial Differential Equations.
Pearson (5th ed.), Upper Saddle River, NJ, 2012.