MATH 461: Fourier Series and Boundary Value Problems

Chapter II: Separation of Variables

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Outline

For much of the following discussion we will use the following 1D heat equation with constant values of c, ρ, K_0 as a model problem:

$$\frac{\partial}{\partial t}u(x,t) = k\frac{\partial^2}{\partial x^2}u(x,t) + \frac{Q(x,t)}{c\rho}, \qquad \text{for } 0 < x < L,\ t > 0$$

with initial condition

$$u(x,0) = f(x) \qquad \text{for } 0 < x < L$$

and boundary conditions

$$u(0,t) = T_1(t), \quad u(L,t) = T_2(t) \qquad \text{for } t > 0$$

Linearity will play a very important role in our work.

Definition

The operator $\mathcal{L}$ is linear if

$$\mathcal{L}(c_1 u_1 + c_2 u_2) = c_1 \mathcal{L}(u_1) + c_2 \mathcal{L}(u_2),$$

for any constants c_1, c_2 and functions u_1, u_2.

Differentiation and integration are linear operations.

Example

- Consider ordinary differentiation of a univariate function, i.e., $\mathcal{L} = \frac{d}{dx}$. Then

$$\frac{d}{dx}(c_1 f_1 + c_2 f_2)(x) = c_1 \frac{d}{dx} f_1(x) + c_2 \frac{d}{dx} f_2(x).$$

Example

- The same is true for partial derivatives:

$$\frac{\partial}{\partial t}\left(c_1 u_1 + c_2 u_2\right)(x,t) = c_1 \frac{\partial}{\partial t} u_1(x,t) + c_2 \frac{\partial}{\partial t} u_2(x,t).$$

- In particular, the heat operator $\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$ is linear. Therefore, the heat equation

$$\frac{\partial}{\partial t} u(x,t) - k \frac{\partial^2}{\partial x^2} u(x,t) = 0$$

  is a linear PDE. If the given right-hand side function is identically zero, then the PDE is called homogeneous.

Remark

A linear homogeneous equation, $\mathcal{L}u = 0$, always has at least the trivial solution $u \equiv 0$.

Example

Are the following equations linear or nonlinear, homogeneous or nonhomogeneous?

- $$\frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) = f(x,y).$$

  is linear and generally nonhomogeneous (Poisson's equation).

- $$\frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) = 0.$$

  is linear and homogeneous (Laplace's equation).

- $$\frac{\partial}{\partial t}u(x,t) - \kappa\frac{\partial}{\partial x}\left[u(x,t)\frac{\partial}{\partial x}u(x,t)\right] = 0.$$

  is nonlinear and homogeneous (nonlinear heat equation, thermal conductivity depends on temperature).

Theorem (Superposition Principle)

If u_1 and u_2 are both solutions of a linear homogeneous equation $\mathcal{L}u = 0$ and c_1, c_2 are arbitrary constants, then $c_1 u_1 + c_2 u_2$ is also a solution of $\mathcal{L}u = 0$.

Proof.

We are given a linear operator $\mathcal{L}$ and functions u_1, u_2 such that

$$\mathcal{L}u_1 = 0, \quad \mathcal{L}u_2 = 0.$$

We need to show that

$$\mathcal{L}(c_1 u_1 + c_2 u_2) = 0.$$

Straightforward computation gives

$$\mathcal{L}(c_1 u_1 + c_2 u_2) \overset{\mathcal{L}\text{ linear}}{=} c_1 \underbrace{\mathcal{L}u_1}_{=0} + c_2 \underbrace{\mathcal{L}u_2}_{=0} = 0.$$

□

We want to solve the PDE

$$\frac{\partial}{\partial t}u(x,t) = k\frac{\partial^2}{\partial x^2}u(x,t), \qquad \text{for } 0 < x < L,\ t > 0 \tag{1}$$

with initial condition

$$u(x,0) = f(x) \qquad \text{for } 0 < x < L \tag{2}$$

and boundary conditions

$$u(0,t) = u(L,t) = 0 \qquad \text{for } t > 0 \tag{3}$$

This is a linear and homogeneous PDE with linear and homogeneous BCs — a perfect candidate for the technique of separation of variables.

Separation of Variables

This technique often just "works", especially for linear homogeneous PDEs and BCs, by magically(?) converting the PDE to a pair of ODEs — and those we should be able to solve[1].

The starting point is to take the unknown function $u = u(x, t)$ and "separate its variables", i.e., to make the *Ansatz*

$$u(x,t) = \varphi(x)G(t) \tag{4}$$

In other words, we just guess that the solution u is of this special form, and hope for the best.

Remark

You may remember another form of separation of variables from MATH 152 or MATH 252 (separable ODEs). In that case the right-hand side of the DE is given with separated variables, i.e., $\frac{dy}{dx} = f(x)g(y)$. *Now we assume (or hope) that the solution is separable.*

[1] If you don't remember, you might want to review Chapters 2 and 5 (maybe also 4) of something like [Zill].

If u is of the form $u(x,t) = \varphi(x)G(t)$ then

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= \varphi(x)\frac{\mathrm{d}}{\mathrm{d}t}G(t)\\ \frac{\partial^2}{\partial x^2}u(x,t) &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi(x)G(t)\end{aligned}$$

Therefore the PDE (1) turns into

$$\varphi(x)\frac{\mathrm{d}}{\mathrm{d}t}G(t) = k\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi(x)G(t)$$

Now we separate variables:

$$\underbrace{\frac{1}{kG(t)}\frac{\mathrm{d}}{\mathrm{d}t}G(t)}_{\text{depends only on } t} = \underbrace{\frac{1}{\varphi(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi(x)}_{\text{depends only on } x}$$

The only way for this equation to be true for all x and t is if both sides are constant (independent of x and t).

Therefore

$$\frac{1}{kG(t)}\frac{\mathrm{d}}{\mathrm{d}t}G(t) = \frac{1}{\varphi(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi(x) = -\lambda \tag{5}$$

The constant λ is known as the separation constant.
The "$-$" sign appears mostly for cosmetic purposes.

Equations (5) give two separate ODEs:

$$\begin{aligned} \varphi''(x) &= -\lambda\varphi(x) && (6)\\ G'(t) &= -\lambda k G(t) && (7) \end{aligned}$$

Before we attempt to solve the two ODEs we note that from the BCs (3) and the Ansatz (4) we get (assuming $G(t) \neq 0$)

$$\begin{aligned} u(0,t) &= \varphi(0)G(t) = 0 \\ &\implies \varphi(0) = 0 \end{aligned} \tag{8}$$

and

$$\begin{aligned} u(L,t) &= \varphi(L)G(t) = 0 \\ &\implies \varphi(L) = 0 \end{aligned} \tag{9}$$

Together, (6), (8), and (9) form a two-point ODE boundary value problem.

Remark

Note that the initial condition, (2), $u(x,0) = f(x)$ *does* not *become an initial condition for* (7)

$$G'(t) = -\lambda k G(t)$$

(since the IC provides spatial, x, information, while (7) *is an ODE in time t).*
Instead, (7) *provides us only with*

$$G(t) = ce^{-\lambda kt}$$

and we will use the initial condition (2) *elsewhere later.*

Solution of the Two-Point BVP

We now solve

$$\begin{aligned} \varphi''(x) &= -\lambda\varphi(x) \\ \varphi(0) &= \varphi(L) = 0. \end{aligned}$$

This kind of problem is discussed in detail in MATH 252 (see, e.g., Chapter 5 of [Zill]).
The characteristic equation of this ODE is

$$r^2 = -\lambda,$$

which is obtained from another *Ansatz*, namely $\varphi(x) = e^{rx}$.
What are the roots r (and therefore the general solution φ)?

For a real separation constant λ there are three cases.
Case I, $\lambda > 0$: In this case, $r^2 = -\lambda$ gives us

$$r = \pm i\sqrt{\lambda}$$

along with the general solution

$$\varphi(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right).$$

Now we use the BCs:

$$\varphi(0) = 0 = c_1 \underbrace{\cos 0}_{=1} + c_2 \underbrace{\sin 0}_{=0} \quad \Longrightarrow \quad c_1 = 0$$

$$\varphi(L) = 0 \overset{c_1=0}{=} c_2 \sin\left(\sqrt{\lambda}L\right) \quad \Longrightarrow \quad c_2 = 0 \text{ or } \sin\sqrt{\lambda}L = 0$$

The solution $c_1 = c_2 = 0$ is not desirable (since it leads to the trivial solution $\varphi \equiv 0$). Therefore, at this point we conclude

$$c_1 = 0 \quad \text{and} \quad \sin\sqrt{\lambda}L = 0.$$

Our conclusions

$$c_1 = 0 \quad \text{and} \quad \sin \sqrt{\lambda} L = 0$$

do not yet specify the solution φ, so we still have work to do.
Note that the equation $\sin \sqrt{\lambda} L = 0$ is true whenever $\sqrt{\lambda} L = n\pi$ for any positive integer n.
In other words, we get

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots$$

Each eigenvalue $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ gives us an eigenfunction

$$\varphi_n(x) = c_2 \sin \frac{n\pi}{L} x, \qquad n = 1, 2, 3, \ldots$$

— each one of which is a solution to the BVP.

Case II, $\lambda = 0$: In this case, $r^2 = -\lambda = 0$ implies $r = 0$ or

$$\varphi(x) = c_1 + c_2 x.$$

The BCs lead to:

$$\varphi(0) = 0 = c_1$$

$$\varphi(L) = 0 \stackrel{c_1=0}{=} c_2 L \quad \Longrightarrow \quad c_2 = 0$$

Now we have only the solution $c_1 = c_2 = 0$, i.e., the trivial solution $\varphi \equiv 0$.
Since by definition $\varphi \equiv 0$ cannot be an eigenfunction, this implies that $\lambda = 0$ is not an eigenvalue for our BVP.
In other words, this case does not contribute to the solution.

Case III, $\lambda < 0$: Now $r^2 = \underbrace{-\lambda}_{>0}$ implies $r = \pm\sqrt{-\lambda}$ or

$$\varphi(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}.$$

The BCs lead to:

$$\varphi(0) = 0 = c_1 + c_2 \quad \Longrightarrow \quad c_2 = -c_1$$

$$\begin{aligned} \varphi(L) = 0 \quad &\stackrel{c_2=-c_1}{=} \quad c_1 e^{\sqrt{-\lambda}L} - c_1 e^{-\sqrt{-\lambda}L} \\ \text{or} \quad & \quad c_1 e^{\sqrt{-\lambda}L} = c_1 e^{-\sqrt{-\lambda}L} \end{aligned}$$

The last row can only be true if $c_1 = 0$ or $L = 0$. The latter does not make any physical sense, so we again have only the trivial solution $c_1 = c_2 = 0$ (or $\varphi \equiv 0$).

Remark

Instead of $\varphi(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ we could have used the alternate formulation $\varphi(x) = c_1 \cosh\left(\sqrt{-\lambda}x\right) + c_2 \sinh\left(\sqrt{-\lambda}x\right)$ — to the same effect.

Summary (so far)

The two-point BVP

$$\begin{aligned}\varphi''(x) &= -\lambda\varphi(x)\\ \varphi(0) &= \varphi(L) = 0\end{aligned}$$

has eigenvalues

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots$$

and eigenfunctions

$$\varphi_n(x) = \sin\frac{n\pi x}{L}, \qquad n = 1, 2, 3, \ldots$$

and together with the solution for G found above we have that…

Summary (cont.)

The PDE-BVP

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) &= k\frac{\partial^2}{\partial x^2}u(x,t), \qquad \text{for } 0 < x < L,\ t > 0 \\ u(0,t) &= u(L,t) = 0 \qquad \text{for } t > 0 \end{aligned}$$

has solutions

$$\begin{aligned} u_n(x,t) &= \varphi_n(x)G_n(t) \\ &= \sin\frac{n\pi x}{L}e^{-\lambda_n kt} \\ &= \sin\frac{n\pi x}{L}e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \qquad n = 1,2,3,\ldots \end{aligned}$$

Remark

- *Note that so far we have not yet used the initial condition* $u(x,0) = f(x)$.
- *Physically, the temperature should decrease to zero everywhere in the rod, i.e.,*

$$\lim_{t\to\infty} u(x,t) = 0.$$

- *We see that each*

$$u_n(x,t) = \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

satisfies this property.

By the principle of superposition any linear combination of u_n, $n = 1, 2, 3, \ldots$, will also be a solution, i.e.,

$$u(x,t) = \sum_n B_n \sin \frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \tag{10}$$

for arbitrary constants B_n is also a solution.

To get a solution u which also satisfies the initial condition we will have to choose the B_ns accordingly.

Notice that the above solution implies

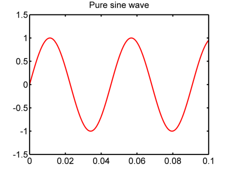
$$u(x,0) = \sum_n B_n \sin \frac{n\pi x}{L}$$

for the initial condition $u(x,0) = f(x)$.

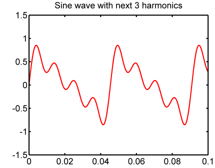
Fourier in Action

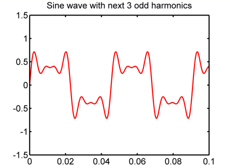
If an air column, a string, or some other object vibrates at a specific frequency it will produce a sound. We illustrate this in the MATLAB script `Soundwaves.m`.

If only one single frequency is present, then we have a sine wave.



Most of the time we hear a more complex sound (with overtones or harmonics). This corresponds to a weighted sum of sine waves with different frequencies.





On March 27, 2008, researchers announced that they had found a sound recording made by Édouard-Léon Scott de Martinville on April 9, 1860 — 17 years before Thomas Edison invented the phonograph.

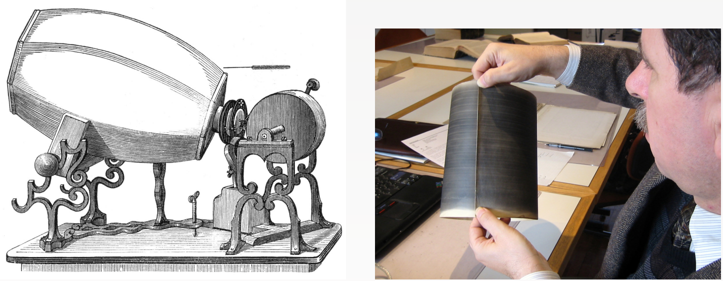


Figure: The phonautograph: a device that scratched sound waves onto a sheet of paper blackened by the smoke of an oil lamp.

Figure: A typical phonautogram.

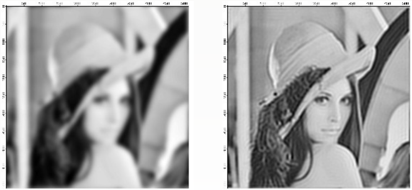
And this is what it sounds like.

Fourier analysis can be used to take apart and analyze complex sounds.

The MATLAB GUI `touchtone` lets us analyze which buttons were pressed on a touch-tone phone.

Wave-like phenomena also play a fundamental role in

- heat flow and other diffusion problems (e.g, the spreading of pollutants),
- vibration problems,
- sound and image file compression (e.g., MP3 or JPEG files),
- filtering of noisy audio or video.



Example

If the initial temperature distribution f is of the form

$$f(x) = \sin \frac{m\pi x}{L},$$

where m is fixed, then

$$u(x,t) = u_m(x,t) = \sin \frac{m\pi x}{L} e^{-k\left(\frac{m\pi}{L}\right)^2 t}$$

will satisfy the entire heat equation problem, i.e., the series solution (10) collapses to just one term, so $B_n = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}.$

Example

If the initial temperature distribution f is of the form

$$f(x) = \sum_{n=1}^{M} B_n \sin \frac{n\pi x}{L},$$

then

$$u(x,t) = \sum_{n=1}^{M} B_n u_n(x,t) = \sum_{n=1}^{M} B_n \sin \frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

will satisfy the entire heat equation problem. In this case, the series solution (10) is finite.

What about other initial temperature distributions f?

The general idea will be to use an infinite series (i.e., a Fourier series) to represent an arbitrary f, i.e., we will show that any (with some mild restrictions) function f can be written as

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

and so the solution of the heat equation (1), (2) and (3) with arbitrary f is given by

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

Remaining question: How do the coefficients B_n depend on f?

Orthogonality (of vectors)

Earlier we noted that the angle θ between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is related to the dot product by

$$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|\|\boldsymbol{b}\|},$$

and therefore the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal, $\boldsymbol{a}\perp\boldsymbol{b}$ (or perpendicular, i.e., $\theta = \frac{\pi}{2}$), if and only if $\boldsymbol{a}\cdot\boldsymbol{b} = 0$.
In terms of the vector components this becomes

$$\boldsymbol{a}\perp\boldsymbol{b} \quad\Longleftrightarrow\quad \boldsymbol{a}\cdot\boldsymbol{b} = \sum_{i=1}^{n} a_i b_i = 0.$$

- If A, B are two sets of vectors then A is orthogonal to B if $\boldsymbol{a}\cdot\boldsymbol{b} = 0$ for every $\boldsymbol{a}\in A$ and every $\boldsymbol{b}\in B$.
- A is an orthogonal set (or simply orthogonal) if $\boldsymbol{a}\cdot\boldsymbol{b} = 0$ for every $\boldsymbol{a}, \boldsymbol{b}\in A$ with $\boldsymbol{a}\neq\boldsymbol{b}$.

Orthogonality (of functions)

We can let our "vectors" be functions, f and g, defined on some interval $[a, b]$. Then f and g are orthogonal on $[a, b]$ with respect to the weight function ω if and only if $\langle f, g\rangle = 0$, where the inner product is defined by

$$\langle f, g\rangle = \int_a^b f(x)g(x)\omega(x)\,\mathrm{d}x.$$

Remark

- *Orthogonality of vectors is usually discussed in linear algebra, while orthogonality of functions is a topic that belongs to functional analysis.*
- *Note that orthogonality of functions always is specified relative to an interval and a weight function.*
- *There are many different classes of orthogonal functions such as, e.g., orthogonal polynomials, trigonometric functions, or wavelets.*
- *Orthogonality is one of the most fundamental (and useful) concepts in mathematics.*

Example

1. Show that the polynomials $p_1(x) = 1$ and $p_2(x) = x$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function $\omega(x) \equiv 1$.
2. Determine the constants α and β such that a third polynomial p_3 of the form
$$p_3(x) = \alpha x^2 + \beta x - 1$$
is orthogonal to both p_1 and p_2.

The polynomials p_1, p_2, p_3 are known as the first three Legendre polynomials.

Solution

Altogether, we need to show that

$$\int_{-1}^{1} p_j(x) p_k(x) \omega(x) \, dx = 0, \qquad \text{whenever } j \neq k = 1, 2, 3$$

Solution (cont.)

1. $p_1(x) = 1$ and $p_2(x) = x$ are orthogonal since

$$\int_{-1}^{1} \underbrace{p_1(x)}_{=1} \underbrace{p_2(x)}_{=x} \underbrace{\omega(x)}_{=1} \, \mathrm{d}x = \int_{-1}^{1} x \, \mathrm{d}x = \left. \frac{x^2}{2} \right|_{-1}^{1} = 0.$$

Of course, we also know that the integral is zero since we integrate an odd function over an interval symmetric about the origin.

Solution (cont.)

2. We need to find α and β such that both

$$\int_{-1}^{1} p_1(x)p_3(x)\omega(x)\,\mathrm{d}x = \int_{-1}^{1} p_2(x)p_3(x)\omega(x)\,\mathrm{d}x = 0.$$

This leads to

$$\int_{-1}^{1} \left(\alpha x^2 + \beta x - 1\right)\,\mathrm{d}x = \left[\alpha\frac{x^3}{3} + \beta\frac{x^2}{2} - x\right]_{-1}^{1} = \frac{2}{3}\alpha - 2 \stackrel{!}{=} 0$$

and

$$\int_{-1}^{1} x\left(\alpha x^2 + \beta x - 1\right)\,\mathrm{d}x = \left[\alpha\frac{x^4}{4} + \beta\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2}\beta \stackrel{!}{=} 0,$$

so that we have $\alpha = 3$, $\beta = 0$ and

$$p_3(x) = 3x^2 - 1.$$

Orthogonality of Sines

We now show that the functions

$$\left\{\sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \sin\frac{3\pi x}{L}, \ldots\right\}$$

are orthogonal on $[0, L]$ with respect to the weight $\omega \equiv 1$.

To this end we need to evaluate

$$\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx$$

for different combinations of integers n and m.

We will discuss the cases $m \neq n$ and $m = n$ separately.

Case I, $m \neq n$: Using the trigonometric identity

$$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

we get

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,\mathrm{d}x = \frac{1}{2}\int_0^L \left[\cos\left((n-m)\frac{\pi x}{L}\right) - \cos\left((n+m)\frac{\pi x}{L}\right)\right]\mathrm{d}x$$

$$\begin{aligned}
&= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\left((n-m)\frac{\pi x}{L}\right) - \frac{L}{(n+m)\pi}\sin\left((n+m)\frac{\pi x}{L}\right)\right]_0^L \\
&= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\left(\underbrace{\sin\underbrace{(n-m)}_{\text{integer}}\pi}_{=0} - \underbrace{\sin 0}_{=0}\right) - \frac{L}{(n+m)\pi}\left(\underbrace{\sin\underbrace{(n+m)}_{\text{integer}}\pi}_{=0} - \underbrace{\sin 0}_{=0}\right)\right] = 0.
\end{aligned}$$

Case II, $m = n$: Using the trigonometric identity

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

we get

$$\int_0^L \sin^2 \frac{n\pi x}{L}\,\mathrm{d}x = \frac{1}{2}\int_0^L \left(1 - \cos\frac{2n\pi x}{L}\right)\mathrm{d}x$$

$$\begin{aligned} &= \frac{1}{2}\left[x - \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right]_0^L \\ &= \frac{1}{2}\left[(L-0) - \frac{L}{2n\pi}\big(\underbrace{\sin 2n\pi}_{=0} - \underbrace{\sin 0}_{=0}\big)\right] = \frac{L}{2} \end{aligned}$$

Orthogonality of Sines

In summary, we have

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{L}{2} & \text{if } m = n \end{cases}$$

and we have established that the set of functions

$$\left\{\sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \sin\frac{3\pi x}{L}, \ldots\right\}$$

is orthogonal on $[0, L]$ with respect to the weight function $\omega \equiv 1$.

Remark

Later we will also use other orthogonal sets such as cosines, or sines and cosines, and other intervals of orthogonality.

We are now finally ready to return to the determination of the coefficients B_n in the solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

of our model problem.
Recall that we assumed that the initial temperature was representable as

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}.$$

Remark

Note that the set of sines above was infinite. This, together with the orthogonality of the sines will allow us to find the B_n.

Start with

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L},$$

multiply both sides by $\sin \frac{m\pi x}{L}$, and integrate wrt. x from 0 to L.

$$\Longrightarrow \quad \int_0^L f(x) \sin \frac{m\pi x}{L} \, \mathrm{d}x = \int_0^L \left[\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \right] \sin \frac{m\pi x}{L} \, \mathrm{d}x.$$

Assume we can interchange integration and infinite summation[2], then

$$\int_0^L f(x) \sin \frac{m\pi x}{L} \, \mathrm{d}x = \sum_{n=1}^{\infty} B_n \underbrace{\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, \mathrm{d}x}_{= \begin{cases} 0 & \text{if } n \neq m, \\ \frac{L}{2} & \text{if } m = n \end{cases}}.$$

Therefore

$$\int_0^L f(x) \sin \frac{m\pi x}{L} \, \mathrm{d}x = B_m \frac{L}{2}$$

[2]This is not trivial! It requires uniform convergence of the series

But

$$\int_0^L f(x) \sin \frac{m\pi x}{L} \, dx = B_m \frac{L}{2}$$

is equivalent to

$$B_m = \frac{2}{L} \int_0^L f(x) \sin \frac{m\pi x}{L} \, dx, \qquad m = 1, 2, 3, \ldots$$

The B_m are known as the Fourier (sine) coefficients of f.

Example

Assume we have a rod of length L whose left end is placed in an ice bath and then the rod is heated so that we obtain a linear initial temperature distribution (from $u = 0^\circ C$ at the left end to $u = L^\circ C$ at the other end). Now, insulate the lateral surface and immerse both ends in an ice bath fixed at $0^\circ C$.

What is the temperature in the rod at any later time t?

This corresponds to the model problem

$$\begin{array}{lll} \text{PDE:} & \dfrac{\partial}{\partial t}u(x,t) = k\dfrac{\partial^2}{\partial x^2}u(x,t), & 0 < x < L,\ t > 0 \\ \text{IC:} & u(x,0) = x, & 0 < x < L, \\ \text{BCs:} & u(0,t) = u(L,t) = 0, & t > 0. \end{array}$$

Solution

From our earlier work we know that

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

This implies that (just plug in $t = 0$)

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}.$$

But we also know that $u(x,0) = x$, so that we have the Fourier sine series representation

$$x = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

or

$$B_n = \frac{2}{L} \int_0^L x \sin \frac{n\pi x}{L} \, dx, \qquad n = 1, 2, 3, \ldots$$

Solution (cont.)

We now compute the Fourier coefficients of $f(x) = x$, i.e.,

$$B_n = \frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L}\,\mathrm{d}x, \qquad n = 1, 2, 3, \ldots$$

Integration by parts (with $u = x$, $\mathrm{d}v = \sin\frac{n\pi x}{L}\,\mathrm{d}x$) yields

$$\begin{aligned}
B_n &= \frac{2}{L}\left[-x\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right]_0^L + \frac{2}{L}\int_0^L \frac{L}{n\pi}\cos\frac{n\pi x}{L}\,\mathrm{d}x \\
&= \frac{2}{L}\left[-L\frac{L}{n\pi}\cos n\pi + 0\right] + \frac{2}{n\pi}\left[\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_0^L \\
&= -\frac{2L}{n\pi}\underbrace{\cos n\pi}_{=(-1)^n} + \frac{2L}{n^2\pi^2}\left(\sin n\pi - \sin 0\right)
\end{aligned}$$

Therefore,

$$B_n = \frac{2L}{n\pi}(-1)^{n+1} = \begin{cases} \frac{2L}{n\pi} & \text{if } n \text{ is odd,} \\ -\frac{2L}{n\pi} & \text{if } n \text{ is even} \end{cases}$$

The solution of the previous example is illustrated in the Mathematica notebook `Heat.nb`.

A 1D Rod with Insulated Ends

We now solve the same PDE as before, i.e., the heat equation

$$\frac{\partial}{\partial t}u(x,t) = k\frac{\partial^2}{\partial x^2}u(x,t), \qquad \text{for } 0 < x < L,\ t > 0$$

with initial condition

$$u(x,0) = f(x) \qquad \text{for } 0 < x < L$$

and new boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \qquad \text{for } t > 0$$

Since the PDE and its boundary conditions are still linear and homogeneous, we can again try separation of variables.
However, since the BCs have changed, we need to go through a new derivation of the solution.

We again start with the *Ansatz* $u(x,t) = \varphi(x)G(t)$ which turns the heat equation into

$$\varphi(x)\frac{\mathrm{d}}{\mathrm{d}t}G(t) = k\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi(x)G(t)$$

Separating variables with separation constant λ gives

$$\frac{1}{kG(t)}\frac{\mathrm{d}}{\mathrm{d}t}G(t) = \frac{1}{\varphi(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi(x) = -\lambda$$

along with the two separate ODEs:

$$\begin{aligned} G'(t) &= -\lambda k G(t) \quad \Longrightarrow \quad G(t) = ce^{-\lambda k t} \\ \varphi''(x) &= -\lambda\varphi(x) \end{aligned} \tag{11}$$

The ODE (11) now will have a different set of BCs. We have (assuming $G(t) \neq 0$)

$$\begin{aligned} \frac{\partial u}{\partial x} u(0,t) &= \varphi'(0) G(t) = 0 \\ &\implies \varphi'(0) = 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u}{\partial x} u(L,t) &= \varphi'(L) G(t) = 0 \\ &\implies \varphi'(L) = 0 \end{aligned}$$

Next, we find the eigenvalues and eigenfunctions. As before, there are three cases to discuss.

Case I, $\lambda > 0$: So $\varphi''(x) = -\lambda\varphi(x)$ has characteristic equation $r^2 = -\lambda$ with roots

$$r = \pm \mathrm{i}\sqrt{\lambda}.$$

We have the general solution (using our *Ansatz* $\varphi(x) = \mathrm{e}^{rx}$)

$$\varphi(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x.$$

Since the BCs are now given in terms of the derivative of φ we need

$$\varphi'(x) = -\sqrt{\lambda}c_1 \sin\sqrt{\lambda}x + \sqrt{\lambda}c_2 \cos\sqrt{\lambda}x$$

and the BCs give us:

$$\varphi'(0) = 0 = -\sqrt{\lambda}c_1 \underbrace{\sin 0}_{=0} + \sqrt{\lambda}c_2 \underbrace{\cos 0}_{=1} \quad \stackrel{\lambda>0}{\Longrightarrow} \quad c_2 = 0$$

$$\varphi'(L) = 0 \stackrel{c_2=0}{=} -\sqrt{\lambda}c_1 \sin\sqrt{\lambda}L \quad \stackrel{\lambda>0}{\Longrightarrow} \quad c_1 = 0 \text{ or } \sin\sqrt{\lambda}L = 0$$

As always, the solution $c_1 = c_2 = 0$ is not desirable ($\rightsquigarrow$ trivial solution $\varphi \equiv 0$). Therefore, we have

$$c_2 = 0 \qquad \text{and} \qquad \sin\sqrt{\lambda}L = 0 \quad \Longleftrightarrow \quad \sqrt{\lambda}L = n\pi.$$

At the end of Case I we therefore have

- eigenvalues

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots$$

- and eigenfunctions

$$\varphi_n(x) = \cos\frac{n\pi}{L}x, \qquad n = 1, 2, 3, \ldots$$

Case II, $\lambda = 0$: Now $\varphi''(x) = 0$

$$\varphi(x) = c_1 x + c_2$$

so that

$$\varphi'(x) = c_1.$$

Both BCs give us:

$$\left.\begin{array}{l}\varphi'(0) = 0 = c_1 \\ \varphi'(L) = 0 = c_1\end{array}\right\} \implies c_1 = 0.$$

Therefore

$$\varphi(x) = c_2 = \text{const}$$

is a solution — in fact, it's an eigenfunction to the eigenvalue $\lambda = 0$.

Case III, $\lambda < 0$: Now $\varphi''(x) = -\lambda\varphi(x)$ again has characteristic equation $r^2 = -\lambda$ with roots $r = \pm\sqrt{-\lambda}$.
But the general solution is (using our *Ansatz* $\varphi(x) = e^{rx}$)

$$\varphi(x) = c_1 \cosh\sqrt{-\lambda}x + c_2 \sinh\sqrt{-\lambda}x.$$

with

$$\varphi'(x) = \sqrt{-\lambda}c_1 \sinh\sqrt{-\lambda}x + \sqrt{-\lambda}c_2 \cosh\sqrt{-\lambda}x.$$

The BCs give us:

$$\varphi'(0) = 0 = \sqrt{-\lambda}c_1 \underbrace{\sinh 0}_{=0} + \sqrt{-\lambda}c_2 \underbrace{\cosh 0}_{=1} \quad \overset{\lambda<0}{\Longrightarrow} \quad c_2 = 0$$

$$\varphi'(L) = 0 \overset{c_2=0}{=} \sqrt{-\lambda}c_1 \sinh\sqrt{-\lambda}L \quad \overset{\lambda<0}{\Longrightarrow} \quad c_1 = 0 \text{ or } \sinh\sqrt{-\lambda}L = 0$$

As always, the solution $c_1 = c_2 = 0$ is not desirable.
On the other hand, $\sinh A = 0$ only for $A = 0$, and this would imply the unphysical situation $L = 0$.
Therefore, Case III does not provide any additional eigenvalues or eigenfunctions.

Altogether — after considering all three cases — we have

- eigenvalues

$$\lambda = 0 \quad \text{and} \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots$$

- and eigenfunctions

$$\varphi(x) = 1 \quad \text{and} \quad \varphi_n(x) = \cos\frac{n\pi}{L}x, \qquad n = 1, 2, 3, \ldots$$

Summarizing, by the principle of superposition

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

will satisfy the heat equation and the insulated ends BCs for arbitrary constants A_n.

Remark

Since $A_0 = A_0 \cos 0 e^0$ *we can also write*

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Finding the Fourier Cosine Coefficients

We now consider the initial condition

$$u(x,0) = f(x).$$

From our work so far we know that

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}$$

and we need to see how the coefficients A_n depend on f.

In HW problem 2.3.6 you should have shown

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{L}{2} & \text{if } m = n \neq 0, \\ L & \text{if } m = n = 0 \end{cases}$$

and therefore the set of functions

$$\left\{1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \cos\frac{3\pi x}{L}, \ldots\right\}$$

is orthogonal on $[0, L]$ with respect to the weight function $\omega \equiv 1$.

To find the Fourier (cosine) coefficients A_n we start with

$$f(x) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L},$$

multiply both sides by $\cos \frac{m\pi x}{L}$, and integrate wrt. x from 0 to L.

$$\Longrightarrow \quad \int_0^L f(x) \cos \frac{m\pi x}{L} \, \mathrm{d}x = \int_0^L \left[\sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} \right] \cos \frac{m\pi x}{L} \, \mathrm{d}x.$$

Again, assuming interchangeability of integration and infinite summation we get

$$\int_0^L f(x) \cos \frac{m\pi x}{L} \, \mathrm{d}x = \sum_{n=0}^{\infty} A_n \underbrace{\int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, \mathrm{d}x}_{= \begin{cases} 0 & \text{if } n \neq m, \\ \frac{L}{2} & \text{if } m = n \neq 0, \\ L & \text{if } m = n = 0 \end{cases}}.$$

By looking at what remains of

$$\int_0^L f(x)\cos\frac{m\pi x}{L}\,\mathrm{d}x = \sum_{n=0}^{\infty} A_n \underbrace{\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,\mathrm{d}x}_{=\begin{cases} 0 & \text{if } n \neq m, \\ \frac{L}{2} & \text{if } m = n \neq 0, \\ L & \text{if } m = n = 0 \end{cases}}$$

in the various cases we get

$$\begin{aligned} A_0 L &= \int_0^L f(x)\,\mathrm{d}x, \\ A_m \frac{L}{2} &= \int_0^L f(x)\cos\frac{m\pi x}{L}\,\mathrm{d}x, \quad m > 0. \end{aligned}$$

But this is equivalent to

$$\begin{aligned} A_0 &= \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x, \\ A_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x, \quad n > 0, \end{aligned}$$

the Fourier cosine coefficients of f.

Remark

Since the solution in this problem with insulated ends is of the form

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \underbrace{e^{-k\left(\frac{n\pi}{L}\right)^2 t}}_{\substack{\to 0 \text{ for } t \to \infty \\ \text{for any } n}}$$

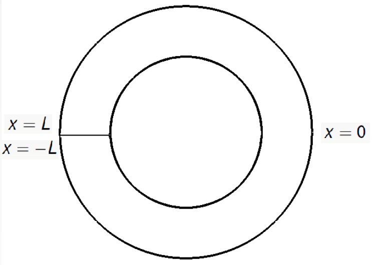
we see that

$$\lim_{t\to\infty} u(x,t) = A_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x,$$

the average of f (cf. our steady-state computations in Chapter 1).

Periodic Boundary Conditions

Let's consider a circular ring with insulated lateral sides.



The corresponding model for heat conduction in the case of perfect thermal contact at the (common) ends $x = -L$ and $x = L$ is

PDE: $$\frac{\partial}{\partial t}u(x,t) = k\frac{\partial^2}{\partial x^2}u(x,t), \qquad \text{for } -L < x < L, \ t > 0$$

IC: $$u(x,0) = f(x) \qquad \text{for } -L < x < L$$

and new periodic boundary conditions

$$\begin{aligned} u(-L,t) &= u(L,t) && \text{for } t > 0 \\ \frac{\partial u}{\partial x}(-L,t) &= \frac{\partial u}{\partial x}(L,t) && \text{for } t > 0 \end{aligned}$$

Everything is again nice and linear and homogeneous, so we use separation of variables.

As always, we use the *Ansatz* $u(x,t) = \varphi(x)G(t)$ so that we get the two ODEs

$$\begin{aligned} G'(t) &= -\lambda k G(t) \quad \Longrightarrow \quad G(t) = ce^{-\lambda k t} \\ \varphi''(x) &= -\lambda \varphi(x), \end{aligned}$$

where the second ODE has periodic boundary conditions

$$\begin{aligned} \varphi(-L) &= \varphi(L), \\ \varphi'(-L) &= \varphi'(L). \end{aligned}$$

Now we look for the eigenvalues and eigenfunctions of this problem.

Case I, $\lambda > 0$: $\varphi''(x) = -\lambda\varphi(x)$ has the general solution

$$\varphi(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x.$$

The BCs are given in terms of both φ and its derivative, so we also need

$$\varphi'(x) = -\sqrt{\lambda}c_1 \sin\sqrt{\lambda}x + \sqrt{\lambda}c_2 \cos\sqrt{\lambda}x.$$

The first BC, $\varphi(-L) = \varphi(L)$, is equivalent to

$$c_1 \underbrace{\cos\sqrt{\lambda}(-L)}_{=\cos\sqrt{\lambda}L} + c_2 \underbrace{\sin\sqrt{\lambda}(-L)}_{=-\sin\sqrt{\lambda}L} \stackrel{!}{=} c_1 \cos\sqrt{\lambda}L + c_2 \sin\sqrt{\lambda}L$$

$$\iff \quad 2c_2 \sin\sqrt{\lambda}L \stackrel{!}{=} 0.$$

While $\varphi'(-L) = \varphi'(L)$, is equivalent to

$$-c_1 \underbrace{\sin\sqrt{\lambda}(-L)}_{=-\sin\sqrt{\lambda}L} + c_2 \underbrace{\cos\sqrt{\lambda}(-L)}_{=\cos\sqrt{\lambda}L} \stackrel{!}{=} -c_1 \sin\sqrt{\lambda}L + c_2 \cos\sqrt{\lambda}L$$

$$\iff \quad 2c_1 \sin\sqrt{\lambda}L \stackrel{!}{=} 0$$

Together, we have

$$2c_1 \sin \sqrt{\lambda} L = 0 \qquad \text{and} \qquad 2c_2 \sin \sqrt{\lambda} L = 0.$$

We can't have both $c_1 = 0$ and $c_2 = 0$. Therefore,

$$\sin \sqrt{\lambda} L = 0 \qquad \text{or} \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

This leaves c_1 and c_2 unrestricted, so that the eigenfunctions are given by

$$\varphi_n(x) = c_1 \cos \frac{n\pi x}{L} + c_2 \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

Case II, $\lambda = 0$: $\varphi''(x) = 0$ implies $\varphi(x) = c_1 x + c_2$ with $\varphi'(x) = c_1$.
The BC $\varphi(-L) = \varphi(L)$ gives us:

$$\begin{aligned} c_1(-L) + c_2 &\stackrel{!}{=} c_1 L + c_2 \\ \iff \quad 2c_1 L &\stackrel{!}{=} 0 \quad \stackrel{L\neq 0}{\Longrightarrow} \quad c_1 = 0. \end{aligned}$$

The BC $\varphi'(-L) = \varphi'(L)$ states

$$c_1 \stackrel{!}{=} c_1$$

which is completely neutral.

Therefore, $\lambda = 0$ is another eigenvalue with associated eigenfunction $\varphi(x) = 1$.

Similar to before, one can establish that Case III, $\lambda < 0$, does not provide any additional eigenvalues or eigenfunctions.

Altogether — after considering all three cases — we have

- eigenvalues

$$\lambda = 0 \quad \text{and} \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots$$

- and eigenfunctions

$$\varphi(x) = 1 \quad \text{and} \quad \varphi_n(x) = c_1 \cos \frac{n\pi}{L} x + c_2 \sin \frac{n\pi}{L} x, \qquad n = 1, 2, 3, \ldots$$

By the principle of superposition

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

is a solution of the PDE with periodic BCs, and the IC $u(x,0) = f(x)$ is satisfied if

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = f(x).$$

In order to find the Fourier coefficients a_n and b_n we need to establish that

$$\left\{1, \cos\frac{\pi x}{L}, \sin\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \sin\frac{2\pi x}{L}, \ldots, \right\}$$

is orthogonal on $[-L, L]$ wrt. $\omega(x) = 1$.

We now look at the various orthogonality relations.

- Since $\int_{-L}^{L}$ even fct $= 2\int_0^L$ even fct we have

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n, \\ L & \text{if } m = n \neq 0, \\ 2L & \text{if } m = n = 0. \end{cases}$$

- Since the product of two odd functions is even we have

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n, \\ L & \text{if } m = n \neq 0. \end{cases}$$

- Since $\int_{-L}^{L}$ odd fct $= 0$, and the product of an even and an odd function is odd, we have

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,\mathrm{d}x = 0.$$

Now, we can determine the coefficients a_n by multiplying both sides of

$$f(x) = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

by $\cos \frac{m\pi x}{L}$, and integrating wrt. x from $-L$ to L.
This gives

$$\begin{aligned}
\int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} \, dx &= \int_{-L}^{L} \left[\sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L} \right] \cos \frac{m\pi x}{L} \, dx \\
&\quad + \underbrace{\int_{-L}^{L} \left[\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \right] \cos \frac{m\pi x}{L} \, dx}_{=0} . \\
\implies \quad a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) \, dx, \\
\text{and} \quad a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx, \quad n \geq 1.
\end{aligned}$$

If we multiply both sides of

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

by $\sin\frac{m\pi x}{L}$, and integrate wrt. x from $-L$ to L
we get

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx &= \underbrace{\int_{-L}^{L}\left[\sum_{n=0}^{\infty} a_n\cos\frac{n\pi x}{L}\right]\sin\frac{m\pi x}{L}\,dx}_{=0} \\ &\quad + \int_{-L}^{L}\left[\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L}\right]\sin\frac{m\pi x}{L}\,dx. \\ \Longrightarrow \quad b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n \geq 1.\end{aligned}$$

Together, a_n and b_n are the Fourier coefficients of f.

Recall that Laplace's equation corresponds to a steady-state heat equation problem, i.e., there are no initial conditions to consider.
We solve the PDE (Dirichlet problem) on a rectangle, i.e.,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 \le x \le L,\ 0 \le y \le H$$

subject to the BCs (prescribed boundary temperature)

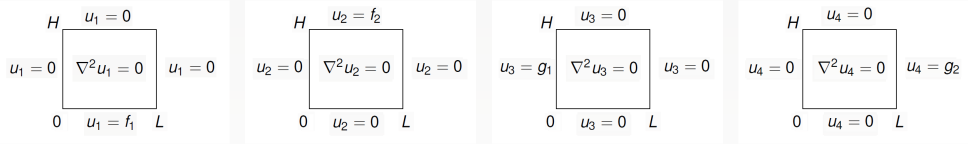
$$\begin{aligned} u(x,0) &= f_1(x), & 0 \le x \le L, \\ u(x,H) &= f_2(x), & 0 \le x \le L, \\ u(0,y) &= g_1(y), & 0 \le y \le H, \\ u(L,y) &= g_2(y), & 0 \le y \le H. \end{aligned}$$

Remark

Note that we can't use separation of variables here since the BCs are not homogeneous!

We can still salvage this approach by breaking the Dirichlet problem up into four sub-problems – each of which has

- one nonhomogeneous BC (similar to how we dealt with the IC earlier),
- and three homogeneous BCs.



We then use the principle of superposition to construct the overall solution from the solutions $u_1, \ldots, u_4$ of the sub-problems:

$$u = u_1 + u_2 + u_3 + u_4.$$

We solve the first problem (the other three are similar):
If we start with the *Ansatz*

$$u_1(x,y) = \varphi(x)h(y)$$

then separation of variables requires

$$\begin{aligned}\frac{\partial^2 u_1}{\partial x^2} &= \varphi''(x)h(y)\\ \frac{\partial^2 u_1}{\partial y^2} &= \varphi(x)h''(y)\end{aligned}$$

Therefore, the Laplace equation becomes

$$\nabla^2 u_1(x,y) = \varphi''(x)h(y) + \varphi(x)h''(y) = 0.$$

We separate

$$\frac{1}{\varphi}\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = -\frac{1}{h}\frac{\mathrm{d}^2 h}{\mathrm{d}y^2} = -\lambda$$

The two resulting ODEs are

- $$\varphi''(x) + \lambda\varphi(x) = 0 \tag{12}$$

  with BCs

  $$u_1(0, y) = 0 \implies \varphi(0) = 0$$
  $$u_1(L, y) = 0 \implies \varphi(L) = 0$$

- and

  $$h''(y) - \lambda h(y) = 0 \tag{13}$$

  with BCs

  $$u_1(x, 0) = f_1(x) \qquad \text{can't use yet}$$
  $$u_1(x, H) = 0 \implies h(H) = 0$$

We solve the ODE (12) as before. Its characteristic equation is $r^2 = -\lambda$, and we study the usual three cases.

Case I, $\lambda > 0$: Then $r = \pm i\sqrt{\lambda}$ and

$$\varphi(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x.$$

From the BCs we have

$$\varphi(0) = 0 = c_1$$

$$\varphi(L) = 0 = c_2 \sin\sqrt{\lambda}L \quad \Longrightarrow \quad \sqrt{\lambda}L = n\pi$$

Thus, our eigenvalues and eigenfunctions (so far) are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad \varphi_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

Case II, $\lambda = 0$: Then $\varphi(x) = c_1 x + c_2$ and the BCs imply

$$\begin{aligned} \varphi(0) = 0 &= c_2 \\ \varphi(L) = 0 &= c_1 L \end{aligned}$$

so that we're left with the trivial solution only.

Case III, $\lambda < 0$: Then $r = \pm\sqrt{-\lambda}$ and

$$\varphi(x) c_1 \cosh \sqrt{-\lambda} + c_2 \sinh \sqrt{-\lambda} x$$

for which the eigenvalues imply

$$\begin{aligned} \varphi(0) = 0 &= c_1 \\ \varphi(L) = 0 &= c_2 \sinh \sqrt{-\lambda} L \quad \Longrightarrow \quad \sqrt{-\lambda} L = 0 \end{aligned}$$

so that we're again only left with the trivial solution.

Now we use the eigenvalues in the second ODE (13), i.e., we solve

$$\begin{aligned} h_n''(y) &= \left(\frac{n\pi}{L}\right)^2 h_n(y) \\ h_n(H) &= 0. \end{aligned}$$

Since $r^2 = \left(\frac{n\pi}{L}\right)^2$ (or $r = \pm\frac{n\pi}{L}$) the solution must be of the form

$$h_n(y) = c_1 e^{\frac{n\pi y}{L}} + c_2 e^{-\frac{n\pi y}{L}}.$$

We can use the BC

$$h_n(H) = 0 = c_1 e^{\frac{n\pi H}{L}} + c_2 e^{-\frac{n\pi H}{L}}$$

to derive

$$c_2 = -c_1 e^{\frac{n\pi H}{L} + \frac{n\pi H}{L}} = -c_1 e^{\frac{2n\pi H}{L}}$$

or

$$h_n(y) = c_1 \left(e^{\frac{n\pi y}{L}} - e^{\frac{n\pi(2H-y)}{L}} \right).$$

Since the second ODE comes with only one homogeneous BC we can now pick the constant c_1 in

$$h_n(y) = c_1 \left(e^{\frac{n\pi y}{L}} - e^{\frac{n\pi(2H-y)}{L}} \right)$$

to get a nice and compact representation for h_n.
The choice

$$c_1 = -\frac{1}{2} e^{-\frac{n\pi H}{L}}$$

gives us

$$\begin{aligned} h_n(y) &= -\frac{1}{2} e^{\frac{n\pi(y-H)}{L}} + \frac{1}{2} e^{\frac{n\pi(H-y)}{L}} \\ &= \sinh \frac{n\pi(H-y)}{L}. \end{aligned}$$

Summarizing our work so far we know (using superposition) that

$$u_1(x,y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}$$

satisfies the first sub-problem except for the nonhomogeneous BC $u_1(x,0) = f_1(x)$.
We enforce this as

$$u_1(x,0) = \sum_{n=1}^{\infty} \underbrace{B_n \sinh \frac{n\pi(H)}{L}}_{=:b_n} \sin \frac{n\pi x}{L} \stackrel{!}{=} f_1(x).$$

From our discussion of Fourier sine series we know

$$b_n = \frac{2}{L}\int_0^L f_1(x) \sin \frac{n\pi x}{L}\,\mathrm{d}x, \qquad n = 1,2,\ldots$$

Therefore

$$B_n = \frac{b_n}{\sinh \frac{n\pi(H)}{L}} = \frac{2}{L \sinh \frac{n\pi(H)}{L}} \int_0^L f_1(x) \sin \frac{n\pi x}{L}\,\mathrm{d}x, \qquad n = 1,2,\ldots$$

Remark

As discussed at the beginning of this example, the solution for the entire Laplace equation is obtained by solving the three similar problems for u_2, u_3 and u_4, and assembling

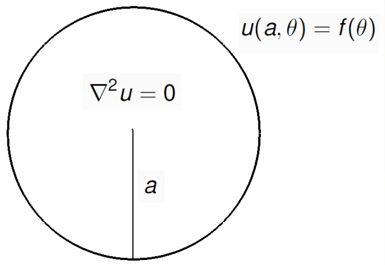
$$u = u_1 + u_2 + u_3 + u_4.$$

The details of the calculations for finding u_3 are given in the textbook [Haberman, pp. 68–71] (where this function is called u_4), and u_4 is determined in [Haberman, Exercise 2.5.1(h)].

Laplace's Equation for a Circular Disk

Now we consider the steady-state heat equation on a circular disk with prescribed boundary temperature.

The model for this case seems to be (using the Laplacian in cylindrical coordinates derived in Chapter 1):



$$\text{PDE:} \qquad \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \text{for } 0 < r < a,\ -\pi < \theta < \pi$$

$$\text{BC:} \qquad u(a,\theta) = f(\theta) \qquad \text{for } -\pi < \theta < \pi$$

Since the PDE involves two derivatives in r and two derivatives in θ we still need three more conditions. How should they be chosen?

Perfect thermal contact (periodic BCs in θ):

$$\begin{aligned} u(r,-\pi) &= u(r,\pi) \qquad \text{for } 0 < r < a \\ \frac{\partial u}{\partial \theta}(r,-\pi) &= \frac{\partial u}{\partial \theta}(r,\pi) \qquad \text{for } 0 < r < a \end{aligned}$$

The three conditions listed are all linear and homogeneous, so we can try separation of variables.

We leave the fourth (and nonhomogeneous) condition open for now.

Remark

This is a nice example where the mathematical model we derive from the physical setup seems to be ill-posed (at this point there is no way we can ensure a unique solution).
However, the mathematics below will tell us how to think about the physical situation, and how to get a meaningful fourth condition.

We begin with the separation *Ansatz*

$$u(r,\theta) = R(r)\Theta(\theta)$$

We can separate our PDE (similar to HW problem 2.3.1)

$$\begin{aligned}
\nabla^2 u(r,\theta) &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \\
\iff & \quad \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}R(r)\right)\Theta(\theta) + \frac{R(r)}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\Theta(\theta) = 0 \\
\iff & \quad \frac{r}{R(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}R(r)\right) = -\frac{1}{\Theta(\theta)}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\Theta(\theta) = \lambda
\end{aligned}$$

Note that λ works better here than $-\lambda$.

The two resulting ODEs are:

- $$\frac{r}{R(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r}R(r) + r\frac{\mathrm{d}^2}{\mathrm{d}r^2}R(r)\right) = \lambda$$
  $$\iff \quad \frac{r^2 R''(r)}{R(r)} + \frac{r}{R(r)}R'(r) - \lambda = 0$$
  or
  $$r^2 R''(r) + rR'(r) - \lambda R(r) = 0. \tag{14}$$

- and
  $$\Theta''(\theta) = -\lambda\Theta(\theta) \tag{15}$$
  for which we have the periodic boundary conditions
  $$\Theta(-\pi) = \Theta(\pi), \qquad \Theta'(-\pi) = \Theta'(\pi).$$

Note that the ODE (15) along with its BCs matches the circular ring example studied earlier (with $L = \pi$).
Therefore, we already know the eigenvalues and eigenfunctions:

$$\lambda_0 = 0, \qquad \lambda_n = n^2, \; n = 1, 2, \ldots$$
$$\Theta_0(\theta) = 1, \qquad \Theta_n(\theta) = c_1 \cos n\theta + c_2 \sin n\theta, \; n = 1, 2, \ldots$$

Using these eigenvalues in (14) we have

$$r^2 R_n''(r) + rR_n'(r) - n^2 R_n(r) = 0, \qquad n = 0, 1, 2, \ldots$$

This type of equation is called a Cauchy-Euler equation (and you should have studied its solution in your first DE course).

We quickly review how to obtain the solution

$$R_n(r) = \begin{cases} c_3 + c_4 \ln r, & \text{if } n = 0, \\ c_3 r^n + c_4 r^{-n}, & \text{for } n > 0. \end{cases}$$

The key is to use the Ansatz $R(r) = r^p$ and to find suitable values of p.

If $R(r) = r^p$, then

$$R'(r) = pr^{p-1} \quad \text{and} \quad R''(r) = p(p-1)r^{p-2},$$

so that the CE equation

$$r^2 R_n''(r) + rR_n'(r) - n^2 R_n(r) = 0$$

turns into

$$\left[p(p-1) + p - n^2\right] r^p = 0$$

Assuming $r^p \neq 0$ we get the characteristic equation

$$p(p-1) + p - n^2 = 0 \quad \Longleftrightarrow \quad p^2 = n^2$$

so that

$$p = \pm n.$$

If $n = 0$, we need to introduce the second (linearly independent) solution $R(r) = \ln r$.

We now look at the two cases.

Case I, $n = 0$: We know the general solution is of the form

$$R(r) = c_3 + c_4 \ln r.$$

We need to find and impose the missing BC.

Note that

$$\ln r \to -\infty \quad \text{for} \quad r \to 0.$$

This would imply that $R(0)$ – and therefore $u(0, \theta)$, the temperature at the center of the disk – would blow up. That is completely unphysical, and we need to prevent this from happening in our model.

We therefore require a bounded temperature at the origin, i.e.,

$$|u(0, \theta)| < \infty \qquad \Longrightarrow \qquad |R(0)| < \infty.$$

This "boundary condition" now implies that $c_4 = 0$, and

$$R(r) = c_3 = \text{const}.$$

Case II, $n > 0$: Now the general solution is of the form

$$R(r) = c_3 r^n + c_4 r^{-n}$$

and we again impose the bounded temperature condition, i.e.,

$$|R(0)| < \infty.$$

Note that

$$\left|r^{-n}\right| = \left|\frac{1}{r^n}\right| \to \infty \qquad \text{as } r \to 0.$$

Therefore this condition implies $c_4 = 0$, and

$$R(r) = c_3 r^n, \qquad n = 1, 2, \ldots$$

Summarizing (and using superposition) we have up to now

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\theta + B_n r^n \sin n\theta.$$

Finally, we use the boundary temperature distribution $u(a,\theta) = f(\theta)$ to determine the coefficients A_n, B_n:

$$u(a,\theta) = A_0 + \sum_{n=1}^{\infty} A_n a^n \cos n\theta + B_n a^n \sin n\theta \stackrel{!}{=} f(\theta).$$

From our earlier work we know that the functions

$$\{1, \cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \ldots\}$$

are orthogonal on the interval $[-\pi,\pi]$ (just substitute $L=\pi$ in our earlier analysis).

It therefore follows as before that

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,\mathrm{d}\theta, \\ A_n a^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,\mathrm{d}\theta, \quad n = 1,2,3\ldots, \\ B_n a^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,\mathrm{d}\theta, \quad n = 1,2,3\ldots, \end{aligned}$$

The solution

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\theta + B_n r^n \sin n\theta$$

of the circular disk problem tells us that the temperature at the center of the disk is given by

$$u(0,\theta) = A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,\mathrm{d}\theta,$$

i.e., the average of the boundary temperature.
In fact, a more general statement is true:

> *The temperature at the center of any circle inside of which the temperature is harmonic (i.e., $\nabla^2 u = 0$) is equal to the average of the boundary temperature.*

This fact is reminiscent of the mean value theorem from calculus and is therefore called the mean value principle for Laplace's equation.

Maximum Principle for Laplace's Equation

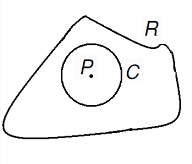
Theorem

Both the maximum and the minimum temperature of the steady-state heat equation on an arbitrary region R occur on the boundary of R.

Proof.

Assume that the maximum/minimum occurs at an arbitrary point P inside of R, and show this leads to a contradiction.

Take a circle C around P that lies inside of R.



By the mean value principle, the temperature at P is the average of the temperature on C.

Therefore, there are points on C at which the temperature is greater/less than or equal the temperature at P.

But this contradicts our assumption that the maximum/minimum temperature occurs at P (inside the circle). □

Well-posedness

Definition

A problem is well-posed if all three of the following statements are true.

- A solution to the problem exists.
- The solution is unique.
- The solution depends continuously on the data (e.g., BCs), i.e., small changes in the data lead to only small changes in the solution.

Remark

This definition was provided by Jacques Hadamard around 1900. Well-posed problems are "nice" problems. However, in practice many problems are ill-posed. For example, the inverse heat problem, i.e., trying to find the initial temperature distribution or heat source from the final temperature distribution (such as when investigating a fire) is ill-posed (see examples below).

Theorem

The Dirichlet problem, $\nabla^2 u = 0$ inside a given region R and $u = f$ on the boundary, is well-posed.

Proof.

(a) Existence: Compute the solution using separation of variables (details depend on the specific domain R).

(b) Uniqueness: Assume u_1 and u_2 are solutions of the Dirichlet problem and show that $w = u_1 - u_2 = 0$, i.e., $u_1 = u_2$.

(c) Continuity: Assume u is a solution of the Dirichlet problem with BC $u = f$ and v is a solution with BC $v = g = f - \varepsilon$ and show that $\min \varepsilon \le u - v \le \max \varepsilon$.

Details for (b) and (c) now follow.

(b) Uniqueness: Assume u_1 and u_2 are solutions of the Dirichlet problem, i.e.,

$$\nabla^2 u_1 = 0, \qquad \nabla^2 u_2 = 0.$$

Let $w = u_1 - u_2$. Then, by linearity,

$$\nabla^2 w = \nabla^2 (u_1 - u_2) = 0.$$

On the boundary, we have for both

$$u_1 = f, \qquad u_2 = f.$$

So, again by linearity,

$$w = u_1 - u_2 = f - f = 0 \quad \text{on the boundary.}$$

(b) (cont.) What does w look like inside the domain?
An obvious inequality is

$$\min(w) \leq w \leq \max(w),$$

for which the maximum principle implies

$$0 \leq w \leq 0 \quad \Longrightarrow \quad w = 0$$

since the maximum and minimum are attained on the boundary (where $w = 0$).

(c) Continuity: We assume

$$\nabla^2 u = 0 \quad \text{and} \quad u = f \text{ on } \partial R,$$
$$\nabla^2 v = 0 \quad \text{and} \quad v = g \text{ on } \partial R,$$

where g is a small perturbation (by the function ε) of f, i.e.,

$$g = f - \varepsilon.$$

Now, by linearity, $w = u - v$ satisfies

$$\nabla^2 w = \nabla^2 (u - v) = 0 \qquad \text{inside } R$$
$$w = u - v = f - g = \varepsilon \qquad \text{on } \partial R.$$

By the maximum principle

$$\min(\varepsilon) \le \underbrace{w}_{=u-v} \le \max(\varepsilon). \qquad \square$$

Example (An ill-posed problem.)

The problem

$$\begin{aligned} \nabla^2 u &= 0 && \text{in } R \\ \nabla u \cdot \hat{\boldsymbol{n}} &= 0 && \text{on } \partial R \end{aligned}$$

does not have a unique solution, since $u = c$ is a solution for any constant c.

Remark

If we interpret the above problem as the steady-state of a time-dependent problem with initial temperature distribution f, then the constant would be uniquely defined as the average of f.

Example (Another ill-posed problem.)

The problem

$$\begin{aligned} \nabla^2 u &= 0 \qquad && \text{in } R \\ \nabla u \cdot \hat{\boldsymbol{n}} &= f \qquad && \text{on } \partial R \end{aligned}$$

may have no solution at all.

The definition of the Laplacian and Green's theorem give us

$$0 = \iint_R \underbrace{\nabla^2 u}_{=0} \, dA \stackrel{\text{def}}{=} \iint_R \nabla \cdot \nabla u \, dA \stackrel{\text{Green}}{=} \int_{\partial R} \underbrace{\nabla u \cdot \hat{\boldsymbol{n}}}_{=f} \, ds,$$

so that only very special functions f permit a solution.

Remark

Physically, this says that the net flux through the boundary must be zero. A non-zero boundary flux integral would allow for a change in temperature (which is unphysical for a steady-state equation).

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