MATH 461: Fourier Series and Boundary Value Problems

Chapter I: The Heat Equation

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Outline

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Physical problem $\longrightarrow$ mathematical model $\longrightarrow$ approximate solution of problem (analytic or numeric)

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Growth of bacteria is often modeled using $\frac{dP}{dt} = kP$. The analytic solution is $P(t) = P_0 e^{kt}$. We could also solve the DE numerically (see MATH 350).

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- may need to truncate infinite series solutions to get practical answer
- possible roundoff or truncation errors in numerical solutions

Physical Problem

Cars traveling on the Chicago highway system.
How long does it take to get from *A* to *B*? How fast are cars able to travel at any specific position and time?

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Mathematical Model

Cars travel on an idealized one-lane road (no on- or off-ramps, no passing) – lots of simplification.

Consider a bunch of cars on our road. For each instance in time, t, each car will be at a specific position $x_i(t)$ moving with velocity $v_i(t)$, and we have

$$v_i(t) = \frac{\mathrm{d}x_i}{\mathrm{d}t}(t), \quad i = 1, \ldots, N.$$

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Then the velocity of a car at position $x(t)$ (starting out at $x(t_0) = x_0$) is given by the solution of the first-order *ODE*

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See the example in the Mathematica notebook `Traffic.nb`.

Refined Mathematical Model

In addition to the velocity field, we now also consider

- the traffic density $\rho(x, t)$, i.e., the number of cars per unit length at any position x and time t [cars/km],
- the traffic flow rate or flux $\phi(x, t)$, i.e., the number of cars per unit time passing at position x and time t [cars/h].

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The two are actually related via the velocity field:

$$\phi(x,t) = \rho(x,t)v(x,t) \qquad \left[\frac{\text{cars}}{\text{h}} = \frac{\text{cars}}{\text{km}}\frac{\text{km}}{\text{h}}\right]$$

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How might we be able to compute one of these, say the density?

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where

$$N(t) = \int_A^B \rho(x,t)\mathrm{d}x. \tag{2}$$

Combining (1) and (2) we get

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Therefore, assuming the density ρ is continuous and A, B are const.,

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This gives rise to the partial differential equation (see Section 12.6 of [Haberman])

$$\frac{\partial}{\partial t}\rho(x,t) = -\frac{\partial}{\partial x}\phi(x,t)$$

$$\text{or} \quad \frac{\partial}{\partial t}\rho(x,t) = -\frac{\partial}{\partial x}\left[\rho(x,t)v(x,t)\right]$$

Even though the traffic flow equation

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is a first-order PDE it is more complicated to study than the second-order equations we will be looking at in this course since the traffic flow equation may develop a discontinuous solution or shock – even for a smooth initial condition (see the image in `Traffic.nb`).

The technique required to solve the traffic flow equation is discussed in MATH 489.

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A nice Java applet simulating traffic flow (including shocks) can be found here and an html5 version here.

Modeling Summary

There are many other kinds of mathematical modeling situations such as

- *data fitting* (e.g., find the best approximation – from a certain linear/nonlinear function class – to given measurement data)
- *parameter estimation* (e.g., find the best parameters for one of the models used earlier – drag coefficient, birth/death rate, etc.)
- *statistical/probabilistic modeling* (e.g., non-deterministic models in finance or weather prediction)
- *discrete modeling* (e.g., determining the best location of a fire department or hospital on a network of roads)
- *geometric modeling* (e.g., used for CAD systems)
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An entertaining overview of the field of mathematical modeling is provided by Charlie's activities on the TV show *NUMB3RS*.

Outline

Heat Flow

We will formulate a model which describes how the temperature u changes over time t in a region (1D $\rightarrow x$, 2D $\rightarrow (x, y)$, or 3D $\rightarrow (x, y, z)$).

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Our models will also require certain initial and boundary conditions such as the entire temperature distribution at the beginning and the temperature on the boundary at any time t.

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We will focus on heat conduction.

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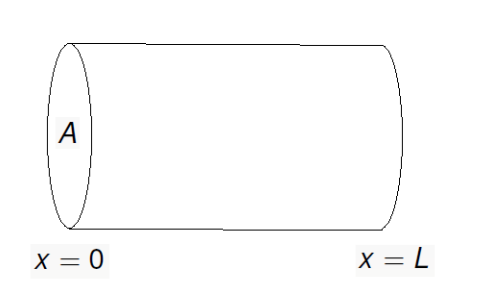
1. We use an energy conservation principle to derive a PDE for the heat energy in a one-dimensional rod.
2. Then we use Fourier's law of heat conduction to relate heat energy to temperature to obtain the so-called heat equation, a PDE that models the temperature in the rod at any position x and time t.

We consider a rod of length L and cross section A



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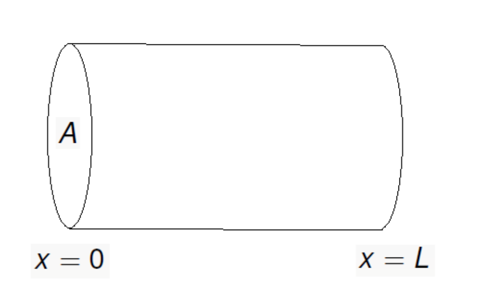

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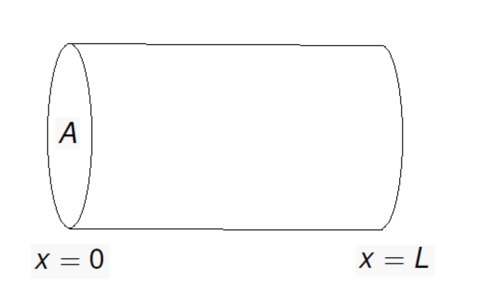

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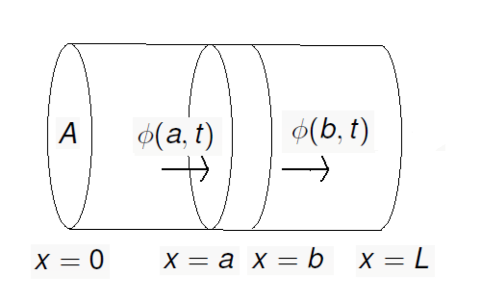
Remark

We assume that e depends only on x and t. This means that the rod is insulated (except possibly at the ends) so that heat can only flow in the x-direction.

Similar to the traffic flow problem we also consider:

- The heat flux or heat flow rate $\phi(x,t)$, i.e., the amount of heat energy per unit time flowing (from left to right) through a unit cross-sectional area at x. Thus, $\phi(x,t) > 0$ denotes flow to right and $\phi(x,t) < 0$ flow to left.



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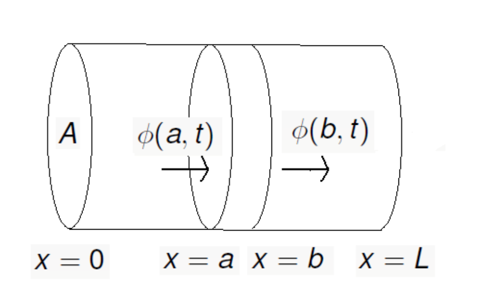

- Possible heat sources $Q(x,t)$, i.e., the amount of heat energy per unit volume generated per unit time.

Conservation of Energy

This is the major physical assumption used:

$$\{\text{Rate of change of heat energy between } x = a \text{ and } x = b\}$$
$$=$$
$$\{\text{rate of heat energy flowing through ends}\}$$
$$+$$
$$\{\text{rate of heat energy generated inside segment of rod}\}$$

Remark

Note that all of these are rates of change per unit of time.

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Rate of heat energy generated inside: $A \int_a^b Q(x,t)\mathrm{d}x$.

Together (conservation of energy – integral form):

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b e(x,t)\mathrm{d}x = \phi(a,t) - \phi(b,t) + \int_a^b Q(x,t)\mathrm{d}x \tag{3}$$

We can further manipulate (3):

First, provided that e is continuous and a, b are const. wrt t,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b e(x,t)\mathrm{d}x = \int_a^b \frac{\partial}{\partial t} e(x,t)\mathrm{d}x.$$

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Second, by the FT of Calculus (provided $\phi \in C^1$),

$$\phi(a,t) - \phi(b,t) = \int_b^a \frac{\partial}{\partial x}\phi(x,t)\mathrm{d}x = -\int_a^b \frac{\partial}{\partial x}\phi(x,t)\mathrm{d}x.$$

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or

$$\int_a^b \left[\frac{\partial}{\partial t}e(x,t) + \frac{\partial}{\partial x}\phi(x,t) - Q(x,t)\right]\mathrm{d}x = 0.$$

Since

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Remark

Equation (3) *is more general than* (4) *since it also applies if* e *and* ϕ *are not continuous.*

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Units:

$$\left[\frac{J}{m}\right] = \left[\frac{J}{kg\,^\circ C}\right]\left[\frac{kg}{m}\right][^\circ C]$$

We can now modify the conservation of energy equation (4)

$$\frac{\partial}{\partial t}e(x,t) = -\frac{\partial}{\partial x}\phi(x,t) + Q(x,t)$$

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Remark

This is still not ideal since it involves both temperature and energy flux. We need to unify further.

Fourier's Law of Heat Conduction

The final physical principle that makes everything come together.

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- Heat flow depends on the specific material of the rod.

The resulting formula is

$$\phi(x,t) = -K_0(x) \frac{\partial}{\partial x} u(x,t),$$

where the thermal conductivity K_0 depends on the material.

Remark

The "$-$" is needed since $\phi > 0$ indicates flow from left to right, but energy also flows from hot to cold and $\frac{\partial}{\partial x} u(x,t) < 0$ if it is warmer on the left.

Using Fourier's law of heat conduction

$$\phi(x,t) = -K_0(x)\frac{\partial}{\partial x}u(x,t),$$

we can rewrite (5) as

$$c(x)\rho(x)\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial}{\partial x}u(x,t)\right] + Q(x,t).$$

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In most cases we will assume c, ρ, K_0 to be constant, i.e., we will use a uniform material. Then we get

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or

$$\frac{\partial}{\partial t}u(x,t) = k\frac{\partial^2}{\partial x^2}u(x,t) + q(x,t),$$

where $k = \frac{K_0}{c\rho}$, the thermal diffusivity and $q(x,t) = \frac{Q(x,t)}{c\rho}$.

Finally, if no sources are present, i.e., $Q(x, t) = 0$, then

$$\frac{\partial}{\partial t}u(x,t) = k\frac{\partial^2}{\partial x^2}u(x,t) \tag{6}$$

is the standard heat equation or diffusion equation.

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Remark

The same form of equation also applies to many other situations, such as diffusion of pollutants, etc.

Outline

Initial Condition

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Remark

As we will see later, one cannot just add any set of conditions. They should be chosen such that the problem is well-posed, i.e., it should allow for the existence of a unique solution that depends continuously on the given conditions.

Boundary Conditions

We will consider three types of boundary conditions.

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Controlled end temperature: e.g., using baths at the ends

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Insulated ends: Since the heat flow is $\phi(x,t) = -K_0 \frac{\partial u}{\partial x}(x,t)$ insulation (i.e, no heat flow) implies $\frac{\partial u}{\partial x} = 0$. Therefore,

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Newton's law of cooling: e.g., cooler air is passed by ends of rod

For Newton's law of cooling we assume there is only partial insulation governed by a positive heat transfer (or convection) coefficient H, e.g.,

$$\phi(0,t) = -H\left[u(0,t) - u_{B_1}(t)\right] \qquad \text{(Newton's law)}$$

Note "$-$" which indicates that – for a rod that is hotter than its environment – heat flow is negative, i.e., flows to the cooler environment (on the "left").

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We also note that

- $H \to 0$ corresponds to perfect insulation
- $H \to \infty$ corresponds to controlled temperature

Outline

We are now ready to solve our first heat equation PDEs. We

- consider different types of boundary conditions
    - fixed end temperature
    - insulated ends

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- consider different types of boundary conditions
  - fixed end temperature
  - insulated ends
- under the fundamental simplifying assumption that we have observed the temperature distribution process for a long time and it has settled down to an equilibrium temperature distribution, i.e., the temperature no longer changes with time.

Controlled End Temperature

Problem:

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= k\frac{\partial^2 u}{\partial x^2}(x,t) \\
u(x,0) &= f(x) \qquad \text{(initial condition)} \\
u(0,t) &= T_1(t) \qquad \text{(left-end BC)} \\
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The IC becomes meaningless[1], and the BCs become

$$u(0) = T_1, \qquad u(L) = T_2.$$

[1] but should be consistent with the BCs

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Therefore

$$u(x) = T_1 + \frac{T_2 - T_1}{L} x,$$

i.e., the temperature distribution interpolates linearly between the fixed end temperatures.

Remark

We will later see that the time dependent PDE problem

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= k\frac{\partial^2 u}{\partial x^2}(x,t) \\ u(x,0) &= f(x) \end{aligned}$$

with time independent BCs

$$u(0,t) = T_1, \qquad u(L,t) = T_2$$

has (in the limit – for very large time) the steady-state solution we just computed, so in this case one can just solve the simple equilibrium problem from the previous slide.

Insulated Boundaries

Now we have

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$$u(x) = C_1 x + C_2.$$

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Remark

- *Note that the ODE problem (7) is not well posed. It does not have a unique solution.*
- *One might expect that the initial temperature distribution $f(x)$ should affect C_2.*
- *In general one should not expect $u(x) = f(x)$, but rather that the initial distribution somehow "levels out".*

Since thermal energy is conserved inside the rod, we can go back to the integral form of the conservation of energy law (3):

$$\frac{d}{dt}\int_0^L e(x,t)dx = \phi(0,t) - \phi(L,t) + \int_0^L Q(x,t)dx$$

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In summary, we get

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i.e., the steady-state temperature distribution is the average of the initial temperature distribution.

Outline

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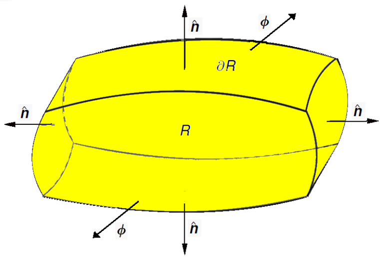

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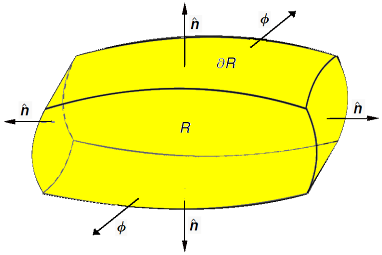

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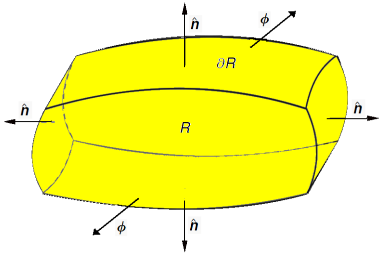
- The unit outer normal vector to R is denoted by $\hat{\boldsymbol{n}}$.



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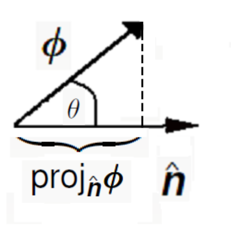

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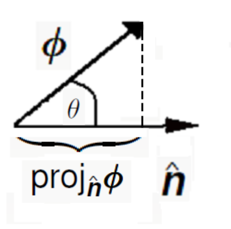

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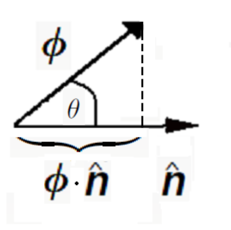

- Relation between angle and dot product:

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Of fundamental importance is also

Theorem (Divergence/Gauss/Ostrogradsky)

Suppose R is a bounded region in $\mathbb{R}^3$ with piecewise smooth boundary ∂R. If $\boldsymbol{f} = (f_1, f_2, f_3) \in C^1$ in an open region that contains R then

$$\iiint_R \nabla \cdot \boldsymbol{f}(x, y, z)\, \mathrm{d}V = \iint_{\partial R} \boldsymbol{f}(x, y, z) \cdot \hat{\boldsymbol{n}}(x, y, z)\, \mathrm{d}S,$$

where $\nabla \cdot \boldsymbol{f} = \frac{\partial}{\partial x} f_1 + \frac{\partial}{\partial y} f_2 + \frac{\partial}{\partial z} f_3 = \operatorname{div} \boldsymbol{f}$ and $\hat{\boldsymbol{n}}(x, y, z)$ is the unit outward normal vector to R at the point (x, y, z) of ∂R.

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Remark

- *This is the 3D-analogue of the FT of Calculus.*
- *In 2D we would be using Green's theorem.*

Conservation of Energy (again)

$$\{\text{Rate of change of heat energy}\}$$

$$=$$

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We need to derive formulas for each one of these three parts.

Total heat energy:

$$\iiint\limits_R e(x,y,z,t)\,\mathrm{d}V = \iiint\limits_R c(x,y,z)\rho(x,y,z)u(x,y,z,t)\,\mathrm{d}V$$

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Using only the normal component of the heat flux (see (8)) we get the rate of heat energy flowing through boundary surface:

$$-\iint_{\partial R} \phi(x, y, z, t) \cdot \hat{\boldsymbol{n}}(x, y, z)\, \mathrm{d}S \tag{11}$$

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Remark

The "$-$" sign appears since outward flow ϕ is positive, but such a flow reduces the heat energy.

Combining (9–11) the conservation of energy principle gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{R} c\rho u \,\mathrm{d}V = -\iint\limits_{\partial R} \phi \cdot \hat{\boldsymbol{n}} \,\mathrm{d}S + \iiint\limits_{R} Q \,\mathrm{d}V \tag{12}$$

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This is where we will use the divergence theorem, i.e.,

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Since this holds for arbitrary R we get (compare with (5))

$$c(x,y,z)\rho(x,y,z)\frac{\partial}{\partial t}u(x,y,z,t) = -\nabla\cdot\phi(x,y,z,t) + Q(x,y,z,t).$$

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Heat equation in 3D:

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i.e.,

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$$\phi \cdot \hat{\boldsymbol{n}} = 0 \quad \iff \quad \nabla u \cdot \hat{\boldsymbol{n}} = 0,$$

i.e., the normal derivative of u is zero.

- Boundary conditions (cont.)
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- *Note that $\hat{\mathbf{n}} = \boldsymbol{i}$ and $\hat{\mathbf{n}} = -\boldsymbol{i}$ correspond to 1D end conditions. For example,*

$$\nabla u \cdot \boldsymbol{i} = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) \cdot (1, 0, 0) = \frac{\partial u}{\partial x}$$

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If in addition $Q = 0$, then we get

Laplace's equation

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- *In 2D these equations look the same, except that we use the 2D Laplacian*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

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$\Longrightarrow$ need to convert the Laplacian to cylindrical and spherical coordinates

Cylindrical Coordinates

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
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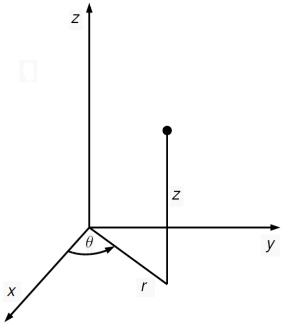
$$



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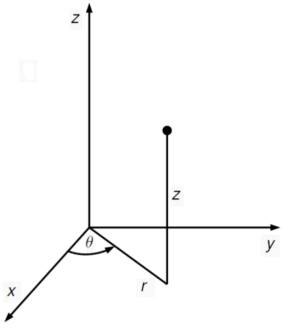

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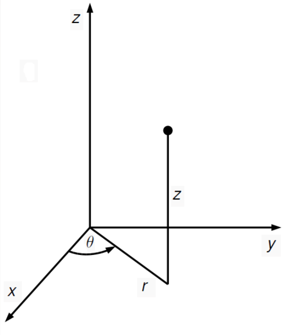

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$$u(x,y,z) = u(r(x,y),\theta(x,y),z).$$

Therefore the derivatives $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2}$ can be expressed using the chain rule.

First calculate

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

First calculate

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \\ &= \frac{\partial u}{\partial r}\frac{2x}{2\sqrt{x^2+y^2}} + \frac{\partial u}{\partial \theta}\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}\end{aligned}$$

First calculate

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \\ &= \frac{\partial u}{\partial r}\frac{2x}{2\underbrace{\sqrt{x^2+y^2}}_{=r}} + \frac{\partial u}{\partial \theta}\underbrace{\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}}_{=\frac{-y}{x^2+y^2}=\frac{-y}{r^2}}\end{aligned}$$

First calculate

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \\
&= \frac{\partial u}{\partial r}\frac{2x}{2\underbrace{\sqrt{x^2+y^2}}_{=r}} + \frac{\partial u}{\partial \theta}\underbrace{\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}}_{=\frac{-y}{x^2+y^2}=\frac{-y}{r^2}} \\
&= \frac{\partial u}{\partial r}\frac{x}{r} - \frac{\partial u}{\partial \theta}\frac{y}{r^2}
\end{aligned}$$

First calculate

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}\frac{2x}{2\underbrace{\sqrt{x^2+y^2}}_{=r}} + \frac{\partial u}{\partial \theta}\underbrace{\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}}_{=\frac{-y}{x^2+y^2}=\frac{-y}{r^2}}$$

$$= \frac{\partial u}{\partial r}\frac{x}{r} - \frac{\partial u}{\partial \theta}\frac{y}{r^2}$$

Using $x = r\cos\theta$ and $y = r\sin\theta$ we get

$$\frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

and so the differential operator

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

Next, using $\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$ three times we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \cos\theta \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)$$

Next, using $\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$ three times we get

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Next, using $\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$ three times we get

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \cos\theta \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right) \\ &= \cos\theta \frac{\partial}{\partial r}\left[\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right] - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left[\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right]\end{aligned}$$

Differentiation using the product rule along with $\frac{\partial^2}{\partial r \partial \theta} = \frac{\partial^2}{\partial \theta \partial r}$ gives us

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \cos\theta \left[\cos\theta \frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta}\right] \\ &\quad - \frac{\sin\theta}{r}\left[-\sin\theta \frac{\partial u}{\partial r} + \cos\theta \frac{\partial^2 u}{\partial r \partial \theta} - \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial \theta^2}\right]\end{aligned}$$

Next, using $\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$ three times we get

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \cos\theta \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right) \\ &= \cos\theta \frac{\partial}{\partial r}\left[\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right] - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left[\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right]\end{aligned}$$

Differentiation using the product rule along with $\frac{\partial^2}{\partial r \partial \theta} = \frac{\partial^2}{\partial \theta \partial r}$ gives us

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \cos\theta\left[\cos\theta\frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta}\right] \\ &\quad - \frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial^2 u}{\partial r \partial \theta} - \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial \theta^2}\right]\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \cos^2\theta\frac{\partial^2 u}{\partial r^2} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} \\ &\quad + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}\end{aligned}$$

We just calculated

$$\frac{\partial^2 u}{\partial x^2} = \cos^2\theta \frac{\partial^2 u}{\partial r^2} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Analogously

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta \frac{\partial^2 u}{\partial r^2} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

We just calculated

$$\frac{\partial^2 u}{\partial x^2} = \cos^2\theta \frac{\partial^2 u}{\partial r^2} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Analogously

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta \frac{\partial^2 u}{\partial r^2} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Therefore

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

We just calculated

$$\frac{\partial^2 u}{\partial x^2} = \cos^2\theta \frac{\partial^2 u}{\partial r^2} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Analogously

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta \frac{\partial^2 u}{\partial r^2} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Therefore

$$\begin{aligned}\nabla^2 u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ &= \left(\cos^2\theta + \sin^2\theta\right)\frac{\partial^2 u}{\partial r^2} + \left(\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}\right)\frac{\partial u}{\partial r} + \left(\frac{\sin^2\theta}{r^2} + \frac{\cos^2\theta}{r^2}\right)\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\end{aligned}$$

We just calculated

$$\frac{\partial^2 u}{\partial x^2} = \cos^2\theta \frac{\partial^2 u}{\partial r^2} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Analogously

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta \frac{\partial^2 u}{\partial r^2} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

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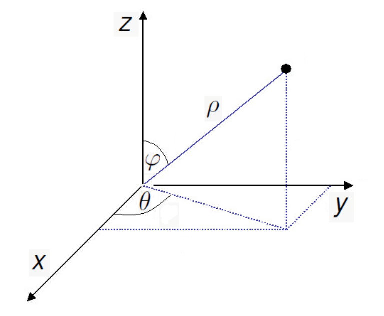
or

Laplacian in cylindrical coordinates:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

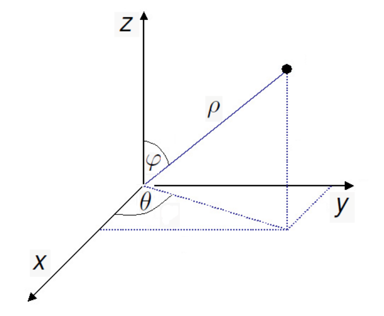
Spherical Coordinates

$$
\begin{aligned}
x &= \rho \sin\varphi \cos\theta \\
y &= \rho \sin\varphi \sin\theta \\
z &= \rho \cos\varphi
\end{aligned}
$$



Spherical Coordinates

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho \cos\varphi \end{aligned}$$



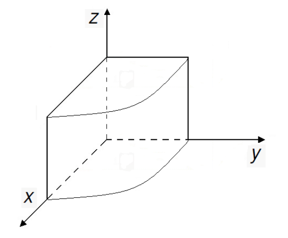
Proceeding similarly as for cylindrical coordinates one can obtain

Laplacian in spherical coordinates:

$$\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left[\rho^2\frac{\partial u}{\partial\rho}\right] + \frac{1}{\rho^2 \sin\varphi}\frac{\partial}{\partial\varphi}\left[\sin\varphi\frac{\partial u}{\partial\varphi}\right] + \frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2 u}{\partial\theta^2}$$

Example

Let $u(r,\theta)$ denote the temperature, independent of z, in a long rod parallel to the z-axis whose cross-section in the xy-plane is given by the circular sector $0 \le r \le 1$, $0 \le \theta \le \frac{\pi}{2}$.

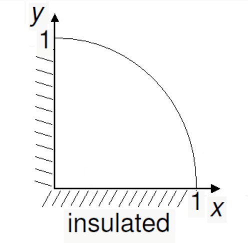


(a) Show

$$\frac{\partial u}{\partial \theta} = -y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}$$

(b) Use the result of (a) to show that if the rod is insulated on its planar surfaces, where $\theta = 0$ and $\theta = \frac{\pi}{2}$, then u must satisfy the boundary conditions

$$\frac{\partial u}{\partial \theta}(r,0) = 0, \quad \frac{\partial u}{\partial \theta}(r,\frac{\pi}{2}) = 0, \quad 0 < r < 1.$$



Solution

(a) We use polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

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Then $u(x,y) = u(x(r,\theta), y(r,\theta))$ and the chain rule gives

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(b) Let $\hat{\boldsymbol{n}}$ be the unit outer normal vector and remember that insulated means that

$$\phi \cdot \hat{\boldsymbol{n}} = 0 \quad \overset{\text{Fourier's law}}{\Longleftrightarrow} \quad -K_0 \nabla u \cdot \hat{\boldsymbol{n}} = 0$$

so that $\nabla u \cdot \hat{\boldsymbol{n}} = 0$, where $\nabla u = (u_x, u_y)$.

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Face $\theta = 0$: Here $\hat{\boldsymbol{n}} = (0, -1)$ (in cartesian coordinates). Therefore

$$\nabla u \cdot \hat{\boldsymbol{n}} = (u_x, u_y) \cdot (0, -1) = -\frac{\partial u}{\partial y}$$

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Now

$$\frac{\partial u}{\partial \theta}(r, 0) \overset{(a)}{=} -y(r, 0)\frac{\partial u}{\partial x}(r, 0) + x(r, 0)\frac{\partial u}{\partial y}(r, 0)$$

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Now

$$\begin{aligned}\frac{\partial u}{\partial \theta}(r, 0) &\overset{(a)}{=} -y(r, 0)\frac{\partial u}{\partial x}(r, 0) + x(r, 0)\frac{\partial u}{\partial y}(r, 0) \\ &= -r \sin\theta|_{\theta=0}\frac{\partial u}{\partial x}(r, 0) + r\cos\theta|_{\theta=0}\frac{\partial u}{\partial y}(r, 0)\end{aligned}$$

(b) Let $\hat{\boldsymbol{n}}$ be the unit outer normal vector and remember that insulated means that

$$\phi \cdot \hat{\boldsymbol{n}} = 0 \quad \overset{\text{Fourier's law}}{\Longleftrightarrow} \quad -K_0 \nabla u \cdot \hat{\boldsymbol{n}} = 0$$

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Now

$$\begin{aligned}
\frac{\partial u}{\partial \theta}(r, 0) &\overset{(a)}{=} -y(r,0)\frac{\partial u}{\partial x}(r,0) + x(r,0)\frac{\partial u}{\partial y}(r,0) \\
&= -\underbrace{r \sin\theta|_{\theta=0}}_{=0} \frac{\partial u}{\partial x}(r,0) + r\cos\theta|_{\theta=0} \underbrace{\frac{\partial u}{\partial y}(r,0)}_{=0\ (\nabla u \cdot \hat{\boldsymbol{n}}=0)}
\end{aligned}$$

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$$\phi \cdot \hat{\boldsymbol{n}} = 0 \quad \overset{\text{Fourier's law}}{\Longleftrightarrow} \quad -K_0 \nabla u \cdot \hat{\boldsymbol{n}} = 0$$

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Now

$$\begin{aligned} \frac{\partial u}{\partial \theta}(r, 0) &\overset{(a)}{=} -y(r,0)\frac{\partial u}{\partial x}(r,0) + x(r,0)\frac{\partial u}{\partial y}(r,0) \\ &= -\underbrace{r \sin\theta|_{\theta=0}}_{=0} \frac{\partial u}{\partial x}(r,0) + r\cos\theta|_{\theta=0} \underbrace{\frac{\partial u}{\partial y}(r,0)}_{=0\ (\nabla u \cdot \hat{\boldsymbol{n}}=0)} \end{aligned}$$

Therefore

$$\frac{\partial u}{\partial \theta}(r, 0) = 0.$$

(b) (cont.)
Face $\theta = \frac{\pi}{2}$: Here $\hat{\boldsymbol{n}} = (-1, 0)$ so that $\nabla u \cdot \hat{\boldsymbol{n}} = -\frac{\partial u}{\partial x}$.

(b) (cont.)
Face $\theta = \frac{\pi}{2}$: Here $\hat{\boldsymbol{n}} = (-1, 0)$ so that $\nabla u \cdot \hat{\boldsymbol{n}} = -\frac{\partial u}{\partial x}$.

Now

$$\frac{\partial u}{\partial \theta}(r, \frac{\pi}{2}) \overset{(a)}{=} -y(r, \frac{\pi}{2})\frac{\partial u}{\partial x}(r, \frac{\pi}{2}) + x(r, \frac{\pi}{2})\frac{\partial u}{\partial y}(r, \frac{\pi}{2})$$

(b) (cont.)
Face $\theta = \frac{\pi}{2}$: Here $\hat{\boldsymbol{n}} = (-1, 0)$ so that $\nabla u \cdot \hat{\boldsymbol{n}} = -\frac{\partial u}{\partial x}$.

Now

$$\begin{aligned}\frac{\partial u}{\partial \theta}(r, \frac{\pi}{2}) &\overset{\text{(a)}}{=} -y(r, \frac{\pi}{2})\frac{\partial u}{\partial x}(r, \frac{\pi}{2}) + x(r, \frac{\pi}{2})\frac{\partial u}{\partial y}(r, \frac{\pi}{2}) \\ &= -r \sin\frac{\pi}{2}\frac{\partial u}{\partial x}(r, \frac{\pi}{2}) + r\cos\frac{\pi}{2}\frac{\partial u}{\partial y}(r, \frac{\pi}{2})\end{aligned}$$

(b) (cont.)

Face $\theta = \frac{\pi}{2}$: Here $\hat{\boldsymbol{n}} = (-1, 0)$ so that $\nabla u \cdot \hat{\boldsymbol{n}} = -\frac{\partial u}{\partial x}$.

Now

$$\begin{aligned}
\frac{\partial u}{\partial \theta}(r, \frac{\pi}{2}) &\overset{(a)}{=} -y(r, \frac{\pi}{2})\frac{\partial u}{\partial x}(r, \frac{\pi}{2}) + x(r, \frac{\pi}{2})\frac{\partial u}{\partial y}(r, \frac{\pi}{2}) \\
&= -r\sin\frac{\pi}{2} \underbrace{\frac{\partial u}{\partial x}(r, \frac{\pi}{2})}_{=0\ (\nabla u \cdot \hat{\boldsymbol{n}} = 0)} + r \underbrace{\cos\frac{\pi}{2}}_{=0} \frac{\partial u}{\partial y}(r, \frac{\pi}{2})
\end{aligned}$$

(b) (cont.)

Face $\theta = \frac{\pi}{2}$: Here $\hat{\boldsymbol{n}} = (-1, 0)$ so that $\nabla u \cdot \hat{\boldsymbol{n}} = -\frac{\partial u}{\partial x}$.

Now

$$\begin{aligned}
\frac{\partial u}{\partial \theta}(r, \frac{\pi}{2}) &\stackrel{(a)}{=} -y(r, \frac{\pi}{2})\frac{\partial u}{\partial x}(r, \frac{\pi}{2}) + x(r, \frac{\pi}{2})\frac{\partial u}{\partial y}(r, \frac{\pi}{2}) \\
&= -r \sin\frac{\pi}{2} \underbrace{\frac{\partial u}{\partial x}(r, \frac{\pi}{2})}_{=0 \ (\nabla u \cdot \hat{\boldsymbol{n}} = 0)} + r \underbrace{\cos\frac{\pi}{2}}_{=0} \frac{\partial u}{\partial y}(r, \frac{\pi}{2})
\end{aligned}$$

Therefore

$$\frac{\partial u}{\partial \theta}(r, \frac{\pi}{2}) = 0.$$

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