MATH 100 – Introduction to the Profession

Proofs

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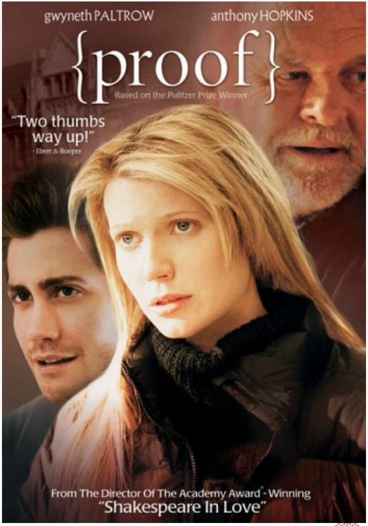
Outline[1]

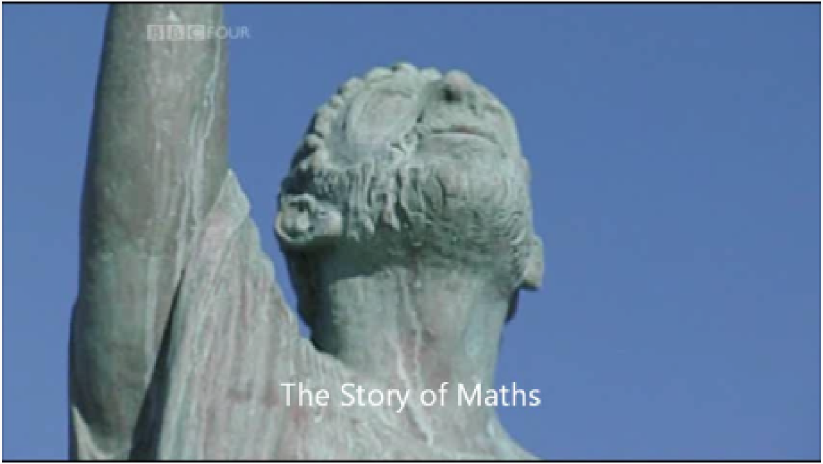
1. Proof
2. Direct Proof
3. Proof by Contradiction
4. Proof by Induction
5. Proof without Words
6. Proofs “From the Book”

[1] Most of this discussion is linked to [Devlin, Section 2.5] and [Gowers, Chapter 3].

“A proof of a statement in mathematics is a logically sound argument that establishes the truth of the statement.” [Devlin]

“Mathematicians . . . demand a proof, that is, an argument that puts a statement beyond all possible doubt.” [Gowers]

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Example

Consider the following problem attributed to Sierpinski:

$$991n^2 + 1 \text{ is not a perfect square.}$$

Is this statement true for all positive integers n?

Try some values:

$$n = 1: \quad 991 \cdot 1 + 1 = 992, \quad \sqrt{992} = 4\sqrt{62} \approx 31.496 \quad \text{T}$$

$$n = 2: \quad 991 \cdot 4 + 1 = 3965, \quad \sqrt{3965} \approx 62.9682 \quad \text{T}$$

$$n = 3: \quad 991 \cdot 9 + 1 = 8920, \quad \sqrt{8920} = 2\sqrt{2230} \approx 94.4458 \quad \text{T}$$

$$n = 10: \quad 991 \cdot 100 + 1 = 99101, \quad \sqrt{99101} \approx 314.803 \quad \text{T}$$

$$n = 537: \quad 991 \cdot 288369 + 1 = 285773680, \quad \sqrt{285773680} \approx 16904.8 \quad \text{T}$$

`http://www.wolframalpha.com/input/?i=Table[Sqrt[991*n^2%2B1]%2C+{n+%2C1%2C1000}]&cdf=1`

Therefore, this statement is *obviously* true.

Not so!

It takes a *looong* time to find a counter-example, but for

$$n = 12055735790331359447442538767$$

we have

$$\begin{aligned} n^2 &= 145340765446276487999885076246978 16\ldots \\ &\quad 6471414204258297880289 \\ 991n^2+1 &= 144032698557259999607886110560755 36\ldots \\ &\quad 2973171476419973199366400 \\ \sqrt{991n^2+1} &= 379516400906811930638014896080 \quad \textbf{F} \end{aligned}$$

Conclusion

Simply checking (many) examples is not good enough to rigorously establish the truth of a statement. We need a mathematical proof.

Theorem (Exercise 2.5.5(e) in [Devlin])

The product of an even and an odd integer is even.

Proof.

To formalize this we assume m is the even integer and n is the odd one. Then the statement we want to prove is

$$(\forall m, n \in \mathbb{Z}) \left[((m \text{ even}) \wedge (n \text{ odd})) \Rightarrow (mn \text{ even}) \right].$$

We can represent

- any even integer as $m = 2k$, for some integer k and
- any odd integer $n = 2\ell + 1$ for some (other) integer ℓ.

Now

$$mn = (2k)(2\ell + 1) = 2(2k\ell + k)$$

and since $2k\ell + k$ is an integer[a] we see that $mn = (2 \times \text{integer})$ is even. $\square$

[a] It doesn't matter if even or odd

As mentioned earlier, proving a statement $\phi \Rightarrow \psi$ directly is difficult. Use of the contrapositive, $(\neg\psi) \Rightarrow (\neg\phi)$, often helps.

Theorem

For all integers n, if n^2 is even then n is even.

Proof.

Here ψ corresponds to "n is even". So we assume that "n is not even", i.e., n is odd.
The theorem is proved if we can show $(\neg\phi)$, i.e., that n^2 is odd.
Any odd number can be represented as $n = 2k + 1$, for some integer k.
Therefore,

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

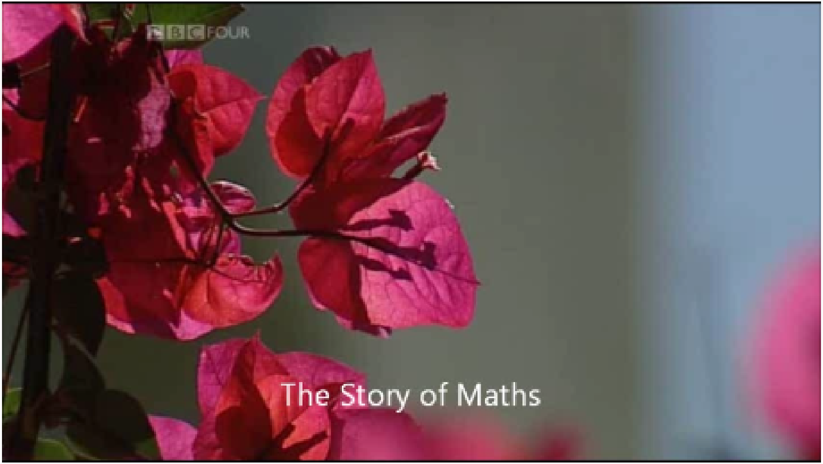
Since $2k^2 + 2k$ is also an integer we have shown that n^2 is odd, and we are done. □

We assume that the conclusion to be proved is false, and argue that this leads to a contradiction.

> *"Reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game." [Hardy]*

Some of the most famous examples of proofs by contradiction are:

- The proof that $\sqrt{2}$ is irrational (probably dating back to Aristotle ca. 350 B.C., see [Devlin, Section 2.5], [Gowers, Chapter 3]).
- The proof that there are infinitely many primes (dating back to Euclid ca. 300 B.C., see below).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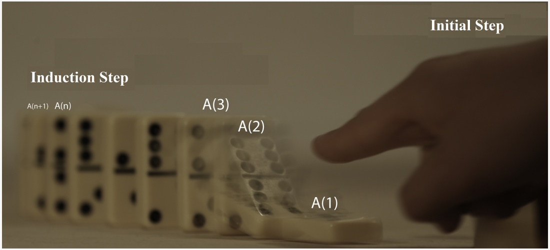


To prove a statement of the form

$$(\forall n \in \mathbb{N})\, A(n)$$

1. Initial step: Show that $A(1)$ holds
2. Induction step: Assume that $A(n)$ holds for an arbitrary n and show that $A(n+1)$ follows, i.e., show

$$(\forall n \in \mathbb{N})\, [A(n) \Rightarrow A(n+1)]$$

3. Combining (1) and (2) we conclude that the statement holds.

This works because of the axioms that define the natural numbers.

Theorem (Exercise 2.5.7(a) in [Devlin], Gauss (9 years old))

For any natural number n, $1+2+3+\ldots+n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Proof

We use mathematical induction to prove $(\forall n \in \mathbb{N})\, A(n)$, where

$$A(n) \text{ stands for } \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

The initial step

$$A(1) \text{ corresponds to } \sum_{k=1}^{1} k = \frac{1(1+1)}{2}.$$

Since both sides of this equality evaluate to one we have ensured that the initial step holds.

Proof cont.

For the induction step we assume that $A(n)$ holds for an arbitrary (but fixed) value of n and try to show that $A(n+1)$ follows.
The left-hand side of $A(n+1)$ is

$$\begin{aligned}
\sum_{k=1}^{n+1} k &= 1+2+3+\ldots+n+(n+1) = \sum_{k=1}^{n} k + (n+1) \\
&\overset{A(n)\text{ holds}}{=} \frac{n(n+1)}{2} + (n+1) \\
&= (n+1)\left(\frac{n}{2}+1\right) = (n+1)\left(\frac{n}{2}+\frac{2}{2}\right) = (n+1)\frac{n+2}{2},
\end{aligned}$$

but this corresponds to the right-hand side of $A(n+1)$.
Since both the initial step and the induction step are true, the statement follows for all $n \in \mathbb{N}$.

Gauss actually proved the above theorem directly (see [Gauss's Day of Reckoning]).
How would such a direct proof go?
Little Gauss had to solve only the problem for $n = 100$:

$$
\begin{array}{ccccccccccccc}
1 & + & 2 & + & 3 & + & \ldots & + & 98 & + & 99 & + & 100 \\
100 & + & 99 & + & 98 & + & \ldots & + & 3 & + & 2 & + & 1 \\
\hline
101 & + & 101 & + & 101 & + & \ldots & + & 101 & + & 101 & + & 101
\end{array}
$$

The number 101 is added 100 times, but we used two copies of the sum we wanted to compute, so

$$1 + 2 + 3 + \ldots + 98 + 99 + 100 = \frac{1}{2} 100 \cdot 101.$$

For general n the argument is analogous:

1	+	2	+	3	+	...	+	(n-2)	+	(n-1)	+	n
n	+	(n-1)	+	(n-2)	+	...	+	3	+	2	+	1
(n+1)	+	(n+1)	+	(n+1)	+	...	+	(n+1)	+	(n+1)	+	(n+1)

and we have

$$1 + 2 + 3 + \ldots + (n-2) + (n-1) + n = \frac{1}{2}n(n+1).$$

This same problem can already be found (with a very similar solution) in [Problems to Sharpen the Young] by the English scholar Alcuin of York written in the 8$^{\text{th}}$ century.

Recall our problem from the beginning of the semester, where we conjectured the following:

Theorem

If the sequence $a_0, a_1, a_2, \ldots$ *satisfies*

$$a_{m+n} + a_{m-n} = \frac{1}{2}\left(a_{2m} + a_{2n}\right) \qquad (*)$$

for all nonnegative integers m *and* n *with* $m \geq n$ *and* $a_1 = 1$*, then* $a_n = n^2$ *for all* $n \in \mathbb{N}_0$*.*

While we computed a number of special values that might serve as the initial step of a mathematical induction proof for this problem, such as

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4, \quad a_3 = 9, \quad \text{and even } a_{2m} = 4a_m,$$

ordinary induction does not suffice for this proof.

Instead we can use strong (or complete) induction. Here the induction step is:

- Assume that for an arbitrary n all of the following statements hold

$$A(1), A(2), \ldots, A(n)$$

  and show that then $A(n+1)$ follows.

So – in contrast to ordinary induction – we now take advantage of complete historical information.

Using the domino analogy, we're using not only the immediate predecessor to knock over the n^{th} domino, but we're allowed to use the combined force of all of its predecessors.

Proof (of sequence problem).

Let $A(n)$ be the statement that $a_n = n^2$.
Certainly the initial step $A(0)$ is true.
Induction step: assume that $A(k)$ is true for all $k = 0, 1, \ldots, m$.
We have (using m and $n = 1$ in $(*)$, and $a_{2m} = 4a_m$ and $a_2 = 4$)

$$a_{m+1} + a_{m-1} = \frac{1}{2}(a_{2m} + a_2) = \frac{1}{2}(4a_m + 4) = 2a_m + 2.$$

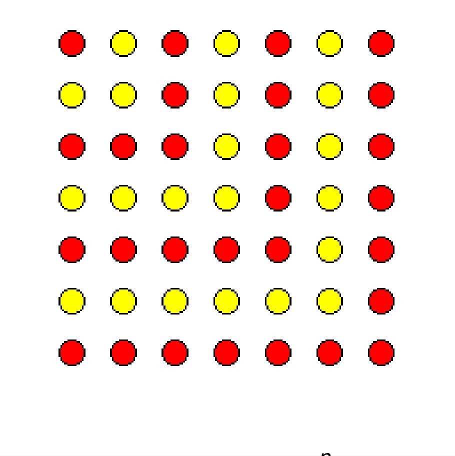
Using our assumption that both $A(m)$ and $A(m-1)$ hold, we get

$$(a_{m+1} + a_{m-1} = 2a_m + 2) \iff \left(a_{m+1} + (m-1)^2 = 2m^2 + 2\right)$$

or

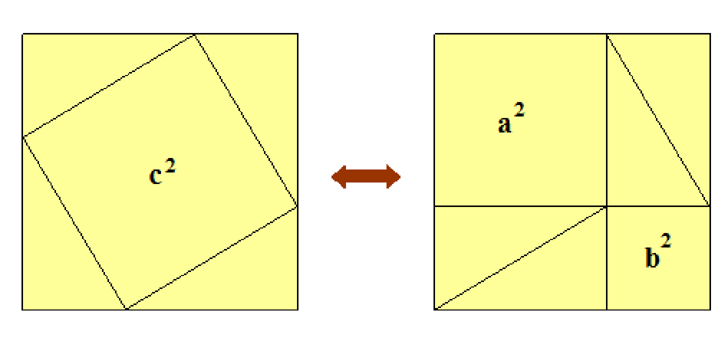
$$a_{m+1} = 2m^2 + 2 - (m^2 - 2m + 1) = m^2 + 2m + 1 = (m+1)^2,$$

which corresponds to $A(m+1)$. $\square$



$$1 + 3 + 5 + \ldots + (2n - 1) = \sum_{k=1}^{n} (2k - 1) = n^2$$

See also HW problem 2.5.8(b) in [Devlin].



$$a^2 + b^2 = c^2$$

See also [Gowers, Chapter 3].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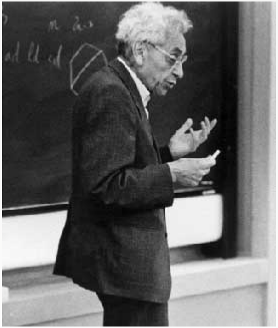
“This one’s from the book.” (Paul Erdős)



Refers to (famous) results with beautiful/elegant proofs.

Example

The Basel problem, first proved by Leonhard Euler in 1735:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

One way to prove this is via Fourier series (see MATH 461).

See [Proofs from THE BOOK] for three different proofs.

Theorem (Book IX, Prop. 20 of Euclid's [Elements])

There are infinitely many primes.

Euclid's Proof (a proof by contradiction).

Assume there are *finitely many* primes: $\{p_1, \ldots, p_r\}$

Now consider the number $n = p_1 p_2 \cdots p_r + 1$.

According to our assumption, n is not a prime number (it's obviously not one of the p_i), so it has prime divisor, say p.

But p is not one of the p_i either since otherwise p would not only be a divisor of n, but also of the product $p_1 p_2 \cdots p_r$.

Consequently, p would be a divisor of the difference $n - p_1 p_2 \cdots p_r = 1$.

But that is impossible, and so we have a contradiction, which means that set $\{p_1, \ldots, p_r\}$ cannot contain all primes. □

The concept of proof is also relevant outside of mathematics.

In [The Elements of a Proposition] the authors analyze some of Abraham Lincoln's speeches as they relate to Euclid's [Elements].

Try this in MATLAB:

```
load penny.mat
contour(P,15)
colormap(copper)
axis ij square
```

Summary

You may see some of these proofs again in classes such as

- MATH 230 – Introduction to Discrete Math
- MATH 410 – Number Theory

Other classes that depend on lots of proofs are

- MATH 332 – Elementary Linear Algebra
- MATH 400 – Real Analysis
- MATH 420 – Geometry
- MATH 430/431 – Applied Algebra I/II
- MATH 453 – Combinatorics
- MATH 454 – Graph Theory

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