MATH 100 – Introduction to the Profession

Modeling and the Exponential Function in MATLAB

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Outline[1]

[1] Most of this discussion is linked to [T. Gowers: Mathematics: A Very Short Introductio Chapter 1] and [ExM, Chapter 8].

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When devising a model, one tries to ignore as much as possible about the phenomenon under consideration, abstracting from it only those features that are essential to understanding its behaviour.

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Earlier Example: Rabbits

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- continuously, so that we obtain a differential equation such as

$$P'(t) = rP(t), \qquad P(0) = P_0,$$

  see below.

Types of Growth

We often differentiate between linear and nonlinear models. In particular, when discussing the growth of a quantity/function/sequence we may encounter

- linear growth of the type
  $P(t) = at$
- polynomial growth of the type
  $P(t) = t^a$
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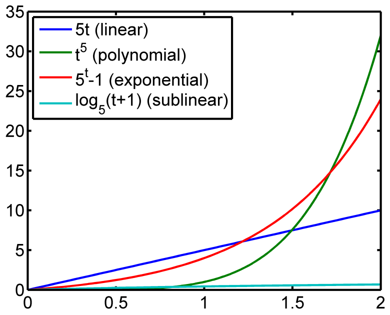

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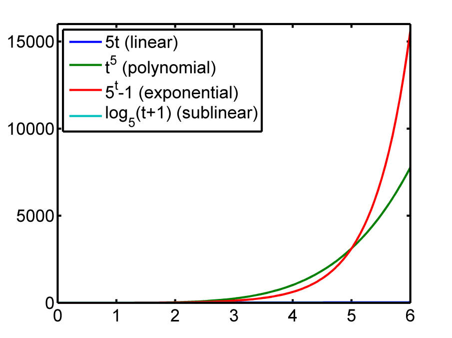

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4. Validate the model. Calibrate parameters if needed by comparing with available data. Make sure the model works for simple/standard situations before applying to something more challenging. Possibly iterate steps 1-4.

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   (b) Desired

   $P(t)$: population at any given time t, with a special interest in large values of t

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   so that we get the specific solution

$$P(t) = P_0 e^{rt}, \qquad \text{for all } t \geq 0.$$

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   It seems virtually impossible to make a prediction of a reasonable population size. Moreover, the growth behavior depends dramatically on (the sign of) the growth rate r.

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   However, for biological populations and long-term predictions we need to rethink/refine our model!

Solving an ODE in MATLAB

We can solve many kinds of ODE initial value problems using, e.g., `ode23()` in MATLAB.
Here's the solution of

$$P'(t) = rP(t), \qquad \text{with } r = 0.3 \text{ and } P_0 = 1000$$

```
r = .3;     % i.e., 30% growth rate
P0 = 1000;    % initial population
tend = 5; % final time for simulation
timespan = [0 tend];    % time interval to simulate
tt = linspace(0,tend,100);    % for plotting
ode23(@(t,P) r*P, timespan, P0)    % MATLAB ODE solver
Pexact = @(t) P0*exp(r*t)     % analytical solution
hold on
plot(tt,Pexact(tt),'r.')
hold off
```

Refined Population Growth Models

For example, we might consider a model in which:

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$$P'(t) = \underbrace{\left(r - r\frac{P(t)}{C}\right)}_{r=r(P)} P(t).$$

  Here C denotes the carrying capacity of the environment.
- This equation can also be solved analytically (a little bit harder, but still basic Calculus), so that we get the solution

$$P(t) = \frac{CP_0e^{rt}}{C + P_0(e^{rt} - 1)}, \qquad \text{for all } t \geq 0,$$

  where P_0 is again the initial population.

MATLAB Solution of Logistic Equation

We solve

$$P'(t) = \left(r - r\frac{P(t)}{C}\right) P(t), \qquad \text{with } r = 1,\ C = 1500 \text{ and } P_0 = 1000$$

again using `ode23()`.

```
r = 1;     % i.e., 100% growth rate
P0 = 1000;    % initial population
tend = 5; % final time for simulation
timespan = [0 tend];    % time interval to simulate
tt = linspace(0,tend,100);    % for plotting
C = 1500;     % capacity
ode23(@(t,P) r*P*(1-P/C), timespan, P0) % MATLAB soln
Pexact = @(t) C*P0*exp(r*t)./(C+P0*(exp(r*t)-1))
hold on
plot(tt,Pexact(tt),'r.')
hold off
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Alternatively, we might consider a model in which:

- The mortality rate depends on the current population, but the birth rate depends on the population that was mature some time earlier.

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- This leads to a delay differential equation of the type

$$P'(t) = -mP(t) + bP(t - \ell),$$

where the mortality rate m and birth rate b are both positive quantities, and time lag ℓ tells us how far to go back to account for maturity[2].

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  where the mortality rate m and birth rate b are both positive quantities, and time lag ℓ tells us how far to go back to account for maturity[2].
- This problem is much harder to solve. For $m = 0$, $b = 1$ and $\ell = 1$, and initial history $P(t) = 1$ for $t \leq 0$ one gets for $t \in [0,5]$

$$P(t) = \begin{cases} t+1 & 0 \leq t \leq 1 \\ \frac{t^2+3}{2} & 1 \leq t \leq 2 \\ \frac{t^3-3t^2+12t+1}{6} & 2 \leq t \leq 3 \\ \frac{t^4-8t^3+42t^2-60t+85}{24} & 3 \leq t \leq 4 \\ \frac{t^5-15t^4+120t^3-430t^2+980t-599}{120} & 4 \leq t \leq 5 \end{cases}$$

[2] Recall the Fibonacci recursion.

MATLAB Solution of DDE

We solve the delay differential equation

$$P'(t) = P(t-1), \qquad \text{with } P(t) = 1 \text{ for } t \le 0$$

using `dde23()`.

```
tend = 5; % final time for simulation
timespan = [0 tend];   % time interval to simulate
tt = linspace(0,tend,100);   % for plotting
lag = 1;  % time delay
hist = 1; % initial history
sol = dde23(@(t,P,d) d, lag, hist, timespan)
plot(sol.x,sol.y)
% analytical solution
Pexact = @(t) (tt>=0 & tt<=1).*(tt+1) + (tt>=1 & tt<=2).*(tt.^2+3)/2 +...
    (tt>=2 & tt<=3).*(tt.^3-3*tt.^2+12*tt+1)/6 +...
    (tt>=3 & tt<=4).*(tt.^4-8*tt.^3+42*tt.^2-60*tt+85)/24 +...
    (tt>=4 & tt<=5).*(tt.^5-15*tt.^4+120*tt.^3-430*tt.^2+980*tt-599)/120
hold on
plot(tt,Pexact(tt),'r.')
hold off
```

Note the piecewise defined solution using logical indexing.

Other refinements of our population growth model might consider

- seasonable variations
- random fluctuations
- partially discrete models, e.g., depending on age groups or gender

Other Types of Problems to Model

- Many engineering problems, such as in mechanics, electronics, or in materials science are described by mathematical models – often involving systems of differential equations describing change.
- Probabilistic/stochastic models are used, e.g., in gambling/games, or for complicated natural or social phenomena such as weather prediction, or financial forecasting.
- Modeling the behavior of gases uses systems of differential equations (Boltzmann equations) to describe the kinetics of the molecules of the gas, but also uses stochastic models since some quantities can only be described in the average sense (due to the Heisenberg uncertainty principle) (read [T. Gowers: Mathematics: A Very Short Introduction, Chapter 1]).
- Modeling of scheduling problems uses techniques of discrete mathematics (such as graphs, and graph coloring), but also optimization algorithms.

Other Types of Problems to Model (cont.)

- Modeling of complex networks in, e.g., biology or neuroscience often requires combinations of many different techniques (discrete, differential equations, probabilistic).
- Logic serves as a tool to model many formal systems, such as in artificial intelligence or formal languages.
- Many other situation in everyday life can be subjected to a mathematical model, e.g., in economics, sociology, politics (voting), etc.

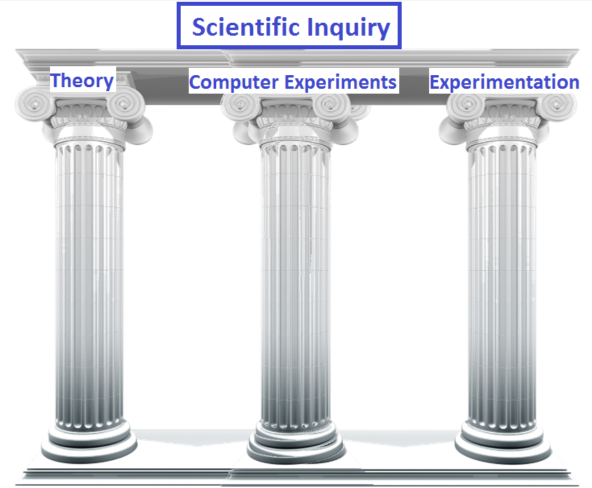
Just about all complex models are simulated using computational techniques. The use of (sophisticated) analytical techniques may greatly improve the efficiency of computational models.

Frequently, we can apply an “abstract” mathematical model to many different practical applications.

The "Third Pillar of Science"

"Together with theory and experimentation, computational science now constitutes the "third pillar" of scientific inquiry, enabling researchers to build and test models of complex phenomena – such as multi-century climate shifts, multidimensional flight stresses on aircraft, and stellar explosions – that cannot be replicated in the laboratory, and to manage huge volumes of data rapidly and economically."

President's Information Technology Advisory Committee [PITAC (2005)]


Scientific Inquiry
Theory
Computer Experiments
Experimentation

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The most basic differentiation rule tells us that any constant multiple $P(t) = c\mathrm{e}^t$ works as well.

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so that $P(t) = c\mathrm{e}^{rt}$ is the general solution of

$$P'(t) = rP(t).$$

◂ Return

Compound interest

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Using $h = 1$, this can also be viewed as[3]

$$A(t+h) = A(t) + rhA(t).$$

[3] Note polynomial growth as for Fibonacci.

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$$\begin{aligned} A(\tfrac{1}{12}) &= A(0) + \tfrac{r}{12}A(0) = (1 + \tfrac{r}{12})A(0) \\ A(\tfrac{2}{12}) &= A(\tfrac{1}{12}) + \tfrac{r}{12}A(\tfrac{1}{12}) = (1 + \tfrac{r}{12})A(\tfrac{1}{12}) = (1 + \tfrac{r}{12})^2 A(0) \\ &\vdots \\ A(t) &= (1 + \tfrac{r}{12})^{12t} A(0) \end{aligned}$$

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Using $h = \frac{1}{12}$, this can again be viewed as[4]

$$A(t+h) = A(t) + rhA(t).$$

[4] Also polynomial growth.

3. If interest is compounded continuously, then we consider

$$A(t+h) = A(t) + rhA(t) \quad \Longleftrightarrow \quad \frac{A(t+h) - A(t)}{h} = rA(t)$$

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3. If interest is compounded continuously, then we consider

$$A(t+h) = A(t) + rhA(t) \quad \Longleftrightarrow \quad \frac{A(t+h)-A(t)}{h} = rA(t)$$

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we get[5]

$$A'(t) = rA(t) \quad \Longrightarrow \quad A(t) = A(0)e^{rt}$$

[5] Now we have exponential growth.

4. For the validation we look at

```
fprintf('            t          annually')
fprintf('        monthly    continuously\n')
format bank
format compact
r = 0.05;
A0 = 10000;
for t = 0:20
   A_annual = (1+r)^t*A0;
   A_month = (1+r/12)^(12*t)*A0;
   A_cont = exp(r*t)*A0;
   disp([t A_annual A_month A_cont])
end
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and see that the models are reasonable.

Return

Student Loan

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3. Each month your payment reduces the current balance, but interest is still added until the loan is paid off. Therefore, following the same line of thought as earlier, after one time period (think $h = \frac{1}{12}$, i.e., one month) the loan amount has been reduced to

$$A(h) = A(0) + rhA(0) - p = (1 + rh)A(0) - p.$$

Continuing month for month we get

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A(h) &= (1+rh)A(0) - p \\
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\end{aligned}
$$

Continuing month for month we get

$$A(h) = (1 + rh)A(0) - p$$

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&\vdots \\
A(nh) &= (1+rh)^n A(0) - p\left((1+rh)^{n-1} + \ldots + (1+rh) + 1\right)
\end{aligned}
$$

Since

$$(1+rh)^{n-1} + \ldots + (1+rh) + 1$$

is a geometric sum we have[6]

$$(1+rh)^{n-1} + \ldots + (1+rh) + 1 = \frac{(1+rh)^n - 1}{(1+rh) - 1} = \frac{(1+rh)^n - 1}{rh}$$

[6] $\sum_{k=1}^{n} q^{k-1} = \frac{q^n - 1}{q - 1}$

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and therefore

$$\begin{aligned} A(nh) &= (1+rh)^n A(0) - p\left((1+rh)^{n-1} + \ldots + (1+rh) + 1\right) \\ &= (1+rh)^n A(0) - p\frac{(1+rh)^n - 1}{rh}. \end{aligned}$$

[6] $\sum_{k=1}^{n} q^{k-1} = \frac{q^n - 1}{q - 1}$

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$$A(nh) = (1+rh)^n A(0) - p\frac{(1+rh)^n - 1}{rh} = 0$$

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4. Without validating the model, we evaluate this using MATLAB:

```
A0 = 20000;
r = .10;
h = 1/12;
n = 36;
p = (1+r*h)^n/((1+r*h)^n-1)*r*h*A0
```

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```

and see that you should make monthly payments of $645.34.

Return