Math 532 — Homework 6 — Due: Wednesday, February 25, 2015

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1. Let $\boldsymbol{x} \in \mathbb{R}^n$, and let A be an $m \times n$ matrix with $\text{rank}(\mathsf{A}) = n$ and let $\mathsf{B} = \mathsf{A}^T\mathsf{A}$. Show that $\|\boldsymbol{x}\|_{\mathsf{B}} = \left(\boldsymbol{x}^T\mathsf{B}\boldsymbol{x}\right)^{1/2}$ is a norm on $\mathbb{R}^n$.

2. (a) Show that $\|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\|_2 \le \sqrt{n}\|\boldsymbol{x}\|_\infty$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

   (b) Why does this imply that if a sequence converges in the ℓ_∞-norm then it converges — to the same limit — also in the ℓ_2-norm? Here a sequence of vectors $\{\boldsymbol{x}_k\} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots\} \subset \mathbb{R}^n$ is said to converge to a limit $\boldsymbol{x}$ in the norm $\|\cdot\|$ if and only if for each $\epsilon > 0$ there exists a positive integer K such that for every $k > K$ we have $\|\boldsymbol{x} - \boldsymbol{x}_k\| < \epsilon$.

   (c) Show that $\|\boldsymbol{x}\|_1 \le \sqrt{n}\|\boldsymbol{x}\|_2$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

   (d) Show that $\|\boldsymbol{x}\|_1 \le n\|\boldsymbol{x}\|_\infty$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

3. Statisticians sometimes like to weight their data. Suppose that $w_1, \ldots, w_n$ are positive scalars (called "weights"). Is $\|\boldsymbol{x}\|_{2,\boldsymbol{w}} = \left(\sum_{i=1}^{n} w_i|x_i|^2\right)^{1/2}$ a norm on $\mathbb{R}^n$? Prove or disprove.

4. Do Exercise 5.1.12 in the textbook. Note that part (b) contains a typo. It should read $\hat{\boldsymbol{y}} = \boldsymbol{y}/\|\boldsymbol{y}\|_q$. Also, do part (c) for the real inner product, i.e., show $|\boldsymbol{x}^T\boldsymbol{y}| \le \|\boldsymbol{x}\|_p\|\boldsymbol{y}\|_q$.

5. Do Exercise 5.1.13 in the textbook.