Math 532 — Homework 3 — Due: Wednesday, February 4, 2015

1. Consider the two matrices

$$\mathsf{A} = \begin{pmatrix} 2 & 2 & 0 & -1 \\ 3 & -1 & 4 & 0 \\ 0 & -8 & 8 & 3 \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} 2 & -6 & 8 & 2 \\ 5 & 1 & 4 & -1 \\ 3 & -9 & 12 & 3 \end{pmatrix}.$$

   (a) Are A and B row equivalent?

   (b) Are A and B column equivalent?

   (c) Are A and B equivalent?

2. Let A be a real $m \times n$ matrix and prove that $\text{rank}(\mathsf{A}) = 1$ if and only if A can be represented as $\mathsf{A} = \boldsymbol{u}\boldsymbol{v}^T$ for some nonzero column vectors $\boldsymbol{u} \in \mathbb{R}^m$, $\boldsymbol{v} \in \mathbb{R}^n$.

3. Consider the upper triangular matrix

$$\mathsf{U} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

   and find permutation matrices P_1 and P_2 so that $\mathsf{P}_1\mathsf{U}\mathsf{P}_2$ is given by the lower triangular matrix

$$\mathsf{P}_1\mathsf{U}\mathsf{P}_2 = \begin{pmatrix} u_{33} & 0 & 0 \\ u_{23} & u_{22} & 0 \\ u_{13} & u_{12} & u_{11} \end{pmatrix}.$$

4. In class we claimed (and used) that the product of (lower) triangular matrices is (lower) triangular. Prove this.

5. Consider the matrix

$$\mathsf{A} = \begin{pmatrix} 2 & 1 & -1 & 3 \\ 4 & 2 & 2 & 7 \\ -2 & -1 & 16 & 2 \\ 8 & 4 & 15 & 24 \end{pmatrix}.$$

   Now compute the following two matrix-matrix products:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ 4 & 1 & 5 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 & 3 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & \frac{15}{4} & 1 & 0 \\ 4 & \frac{19}{4} & \frac{29}{5} & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 & 3 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

   What do you notice? Explain why this does or does not conflict with what we discussed in class.

6. Let $\mathsf{A} = \begin{pmatrix} 1 & 4 & 5 \\ 4 & 18 & 26 \\ 3 & 16 & 30 \end{pmatrix}$.

   (a) Compute the LU factorization of A.

(b) Use the LU factorization to solve the two linear systems $\mathsf{A}\boldsymbol{x}_1 = \boldsymbol{b}_1$ and $\mathsf{A}\boldsymbol{x}_2 = \boldsymbol{b}_2$ with

$$\boldsymbol{b}_1 = \begin{pmatrix} 6 \\ 0 \\ -6 \end{pmatrix}, \qquad \boldsymbol{b}_2 = \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix}.$$

(c) Use the LU factorization to compute A^{-1}.

7. Consider the vector space $\mathcal{V} = \mathbb{R}^{n\times n}$ of square real matrices. Show that the sub*set* $\mathcal{S}$ of upper triangular matrices is also a sub*space* of $\mathcal{V}$.