Randomness: what is that and how to cope with it
(with view towards financial markets)

Igor Cialenco
Dep of Applied Math, IIT
igor@math.iit.etu

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Randomness is almost everywhere

Modeling it (the randomness) is FUN
What’s randomness

- **Event(s) with Random Outcomes**
  Random, Stochastic, Uncertain, Chaotic, Unpredictable

- **Examples of Random Events:**
  flip a coin, temperature next Friday at noon, Dow Jones Industrial Average Tomorrow at 3:40pm, moving of a car in traffic, etc

- **Deterministic Outcomes:** - flipped coin, temp yesterday, number of days in a year 2089, etc

- “Almost Random” - small noise in deterministic system
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  "Probability, science originated in consideration of games of choice, should become the most important object of human knowledge"

  Pierre Simon, Marquis de Laplace, 23 April 1749 - 5 March 1827, France
What is random and what is not

- More a philosophical question
  - causality, predetermined/unknown future, all odds are known/unknown
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  weather in Chicago, spam of predicting the market, the most talented
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  type of randomness
  **gambling** - the rules are known, the sources of randomness are known
  **stock market** - the risk and randomness are changing, the rules and factors are
  unknown, we can only assume something about the randomness (the distribution
  of uncertainty)
- An attempt to describe various types of randomness “The Black Swan” by N.N.Taleb;

- David Aldous book review
  http://www.stat.berkeley.edu/~aldous/157/Books/taleb.html

- Andrew Gelman book review
What parts of mathematics study randomness?
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... and what’s the difference?
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- Probability
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... and what’s the difference?

Both study the same objects and phenomena, but from very different points of view.

... an example will help to see the difference
Flip a coin

- The outcomes Head (H) or Tail (T)
Simplest example

**Flip a coin**

- The outcomes Head (H) or Tail (T)
- Chances of H and T
  
  (a) say equal, 50/50, fair coin
  
  or (b) $\mathbb{P}(H) = p$, $\mathbb{P}(T) = 1 - p$, for some fixed and known $p \in (0, 1)$

This is a probabilistic model of flipping a coin.
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**Probability Theory** assumes the coin (the distribution) is known, and tries to find/predict/study something about future observed events.

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**Problem:** you play a game in which you are paid $5 if H and $3 if T. How much should you pay to enter the game?
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**Problem:** you play a game in which you are paid $5 if H and $3 if T. How much should you pay to enter the game?

**Answer:** In a fair game you should pay the expected winning sum

$$\mathbb{E}({\text{payoff}}) = 5 \cdot p + 3 \cdot (1 - p)$$
Flip a coin ... con’t

- The model is done

- You can find about anything related to this model

  Flip the coin many times, look at the number of heads, number of consecutive heads, first time you have $N$ heads and $M$ tails, etc. All these probabilities can be evaluated.
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Some of the quantities of interest can be found by probabilistic methods (using in particular combinatorics) or by simulations

You do not need a coin to simulate the game (computer can do)

Computer Simulated Outcomes for flipping a coin

- $p = 0.7$: H H T H H H T H H H T T H H H
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$$\hat{p} = \frac{\text{# of Heads}}{\text{# of total observations}}$$

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Statistics - based on past observations we try to find/infer/estimate the probabilities of some events to happen. We try to make sense of past data.
Estimation of probability of getting Head in a loaded coin

Number of observations
Roll a die and get paid the face value

The model: six faces, six outcomes \( \Omega = \{1, 2, 3, 4, 5, 6\} \). Each face ends up with some probability \( p_1, p_2, \ldots, p_6 \). Note \( p_1 + p_2 + \ldots + p_6 = 1 \).
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Fair value to enter the game? Expected payoff
$E(\text{payoff}) = 1 \cdot p_1 + 2 \cdot p_2 + \ldots + 6 \cdot p_6$

Fair die, then $p_1 = p_2 = \ldots = p_6 = 1/6$ and $E(\text{payoff}) = 3.5$

Simulations 2 3 5 5 2 2 3 3 1 2 6 2 1 3 4 5 6 2 2 4 5 6 2 3 1

Other Casino type games. Same idea, as long as the rules are known.
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Roulette? Easy, a fair die with 36 faces
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**Other Casino type games.** Same idea, as long as the rules are known.

- Roulette? Easy, a fair die with 36 faces
- Blackjack? Also “easy”, just more complicated combinatorics. No independency, so one can count the cards
Back to financial markets

predicting the stock price
What is so different in financial markets?

- The rules, sources of randomness, and sources of risk are changing.
- The factors driving the randomness in the market are unknown; we can only assume some properties about them (e.g. distribution).
- The stock price today already reflects all the past information. The price is based on demand and supply.
- Nobody can predict (with certainty) the future stock price.
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HOWEVER!
still many things can be done
Fundamental Law

No Arbitrage or No Free Lunch
(can not make money for sure out of nothing)
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Example (of arbitrage):
Bank ABC: deposit at 3.5% and borrow at 3.8% per year
Bank XYZ: deposit at 3% and borrow at 3.4% per year

Arbitrage: borrow, say $10,000 from XYZ, and deposit into ABC. This costs $0 at initiation. Close out the position at the end of the year, and get a sure profit of $10.
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Disclaimer: of course, we assumed that ABC and XYZ will not default within one year
Hedging/Replication of derivative contract

- Bank PQR wants to buy today the following (future) contract: for no $’s down today, to agree on a price of $\$K$, paid in one year, for getting one share of AAPL (Apple Inc) also in one year.

- Bank KLM wants to sell this contract. Assume that KLM has access to credit (can borrow) for 3.0\% per year.

**Question:** What is $\$K$ that KLM wants to charge PQR?

**Answer:** The fair price $K = $563.4718.
Hedging/Replication of derivative contract

\[ K = \text{‘AAPL price today’ } \times (1 + 0.03) = 547.06 \times 1.03 = 563.4718 \]

- Why? Because KLM can replicate. Assume that KLM enters the contract.
  - Borrow $547.06 for one year under 3%
  - Buy one share of AAPL
  - Zero cost today
  - In one year ....
  - Get \( K = 563.4718 \) from PQR in exchange for that share of AAPL
  - Return to the lender exactly $563.4718 (which is initial borrowing of $547.06 plus the interest of $16.4118)
Idea: Stock price - a banking account, but random (why not?)
Banking account $B_t = B_0 \cdot e^{rt}$, with $r$ - interest rate

$$B_{t+\Delta t} = B_t e^{r \Delta t}$$

Stock - a random banking account, kind of ...

$$S_{t+\Delta t} = S_t e^{\mu \Delta t \pm \sigma \sqrt{\Delta t}}$$

with equal probabilities up or down ($\pm$).
Parameters $\mu, \sigma$ implied from the market or estimated historically.
Simulation of stock price using Black-Scholes-Merton model.
What if the rules are unknown? What if ‘the die’ is changed, and ‘the casino’ does NOT tell us that?
Modeling randomness in real life

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Examples: financial markets, temperature anomalies, turbulence etc
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How to model?
Modeling randomness in real life

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How to model?

- Make simplifications
- Start from simple
- Keep track of general rules and laws of ‘nature’
- Use past data, but do not overuse it
- If no explicit solution, simulation usually helps
Thank You!

The end of the talk ... but not of the story