Math 577 — Homework Assignment 6, due Nov.28, 2006

1. Consider one step of the "pure" QR algorithm applied to a tridiagonal symmetric matrix $A \in \mathbb{R}^{m\times m}$.

   (a) If only eigenvalues are desired, then only $A^{(k)}$ is needed at step k, not $\underline{Q}^{(k)}$. Determine how many flops are required to get from A^{k-1} to $A^{(k)}$ using standard methods studied in class.

   (b) If all the eigenvectors are desired, then the matrix $\underline{Q}^{(k)} = Q^{(1)}Q^{(2)}\cdots Q^{(k)}$ will need to be accumulated too. Determine how many flops are now required to get from step $k-1$ to step k.

2. Show that the basic fixed-point iteration

$$M\boldsymbol{x}^{(k)} = N\boldsymbol{x}^{(k-1)} + \boldsymbol{b}$$

   is equivalent to the following three steps:

   Given $\boldsymbol{x}^{(k-1)}$

   (i) compute the residual $\boldsymbol{r}^{(k-1)} = \boldsymbol{b} - A\boldsymbol{x}^{(k-1)}$,

   (ii) solve $M\boldsymbol{z}^{(k-1)} = \boldsymbol{r}^{(k-1)}$ for $\boldsymbol{z}^{(k-1)}$,

   (iii) define $\boldsymbol{x}^{(k)} = \boldsymbol{x}^{(k-1)} + \boldsymbol{z}^{(k-1)}$.

3. Using the notation of the previous problem, show that

$$\begin{aligned} \boldsymbol{r}^{(k)} &= NM^{-1}\boldsymbol{r}^{(k-1)} \\ \boldsymbol{z}^{(k)} &= M^{-1}N\boldsymbol{z}^{(k-1)}. \end{aligned}$$

4. Find the explicit form of the iteration matrix $G = M^{-1}N$ in the Gauss-Seidel method when

$$A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}.$$