Math 577 — Homework Assignment 1, due Sept.14, 2006

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1. Fact: For any nonsingular $m \times m$ matrix A the j-th unit vector $\boldsymbol{e}_j$ can be expressed in terms of the entries of A and its inverse, i.e.,

$$\boldsymbol{e}_j = \sum_{i=1}^{m} A^{-1}(i,j)A(:,i).$$

   We say that a square or rectangular matrix R with entries $R(i,j)$ is *upper-triangular* if $R(i,j) = 0$ for $i > j$. By considering what space is spanned by the first n columns of R and using the fact above, show that if R is a nonsingular $m \times m$ upper-triangular matrix, then R^{-1} is also upper-triangular.
   Note: The analogous result holds for lower-triangular matrices.

2. The Pythagorean theorem asserts that for a set of n orthogonal vectors $\{\boldsymbol{x}_i\}$,

$$\left\| \sum_{i=1}^{n} \boldsymbol{x}_i \right\|^2 = \sum_{i=1}^{n} \|\boldsymbol{x}_i\|^2.$$

   (a) Prove this in the case $n = 2$ by an explicit computation of $\|\boldsymbol{x}_1 + \boldsymbol{x}_2\|^2$.
   (b) Show that this computation also establishes the general case, by induction.

3. Let $A \in \mathbb{C}^{m \times m}$ be Hermitian. An eigenvector of A is a nonzero vector $x \in \mathbb{C}^m$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$ for some $\lambda \in \mathbb{C}$, the corresponding eigenvalue.

   (a) Prove that all eigenvalues of A are real.
   (b) Prove that if $\boldsymbol{x}$ and $\boldsymbol{y}$ are eigenvectors corresponding to distinct eigenvalues, then $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal.

4. What can be said about the eigenvalues of a unitary matrix?

5. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are m-vectors, the matrix $A = I + \boldsymbol{u}\boldsymbol{v}^*$ is known as a *rank-one perturbation of the identity*. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha\boldsymbol{u}\boldsymbol{v}^*$ for some scalar α, and give an expression for α. For what $\boldsymbol{u}$ and $\boldsymbol{v}$ is A singular? If it is singular, what is null(A)?

6. Read Section 1.4 in the classnotes (Sections 2.1 and 2.2 in Kincaid/Cheney or Lecture 13 in Trefethen/Bau contain similar information).

7. If $\frac{1}{10}$ is correctly rounded to the normalized binary number $(1.a_1a_2\dots a_{23})_2 \times 2^m$, what is the roundoff error? What is the relative roundoff error?

8. In solving the quadratic equation $ax^2 + bx + c = 0$ by use of the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   there is a loss of significance when $4ac$ is small relative to b^2 because then

$$\sqrt{b^2 - 4ac} \approx |b|.$$

   Suggest a method to circumvent this difficulty.

9. Arrange the following formulas in order of merit for computing $\tan x - \sin x$ when x is near 0.

   (a) $\sin x[(1/\cos x) - 1]$,

   (b) $\frac{1}{2}x^3$,

   (c) $(\sin x)/(\cos x) - \sin x$,

   (d) $(x^2/2)(1 - x^2/12)\tan x$,

   (e) $\frac{1}{2}x^2 \tan x$,

   (f) $\tan x \sin^2 x/(\cos x + 1)$.

10. If at most 2 bits of precision are to be lost in the computation of $y = \sqrt{x^2+1} - 1$, what restriction must be placed on x?