Math 350 — Homework Assignment 2, due Feb. 3, 2011

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1. Consider the linear system

$$\begin{aligned} 3x_1 + 4x_2 + 3x_3 &= 16 \\ x_1 + 5x_2 - x_3 &= -12 \\ 6x_1 + 3x_2 + 7x_3 &= 102. \end{aligned}$$

   (a) Solve the system using basic Gaussian elimination (LU decomposition) **without** pivoting. Make sure you follow the algorithm as described in the notes. Do **not** use any "shortcuts" that a clever human might recognize to simplify his/her work.
   Use four significant digits and show all steps of your work.

   (b) What are the factors L and U?

2. Show that the system of equations

$$\begin{aligned} x_1 + 4x_2 + \alpha x_3 &= 6 \\ 2x_1 - x_2 + 2\alpha x_3 &= 3 \\ \alpha x_1 + 3x_2 + x_3 &= 5 \end{aligned}$$

   possesses a unique solution when $\alpha = 0$, no solution when $\alpha = -1$, and infinitely many solutions when $\alpha = 1$. Also, investigate the corresponding situation when the right-hand side is replaced by all zeros.

3. Show how Gaussian elimination **with partial pivoting** works on the system of Problem 1. Show all steps of your work. What do the matrices P, L and U look like?

4. (a) Describe in words what the results of applying the *elementary matrices*

$$\mathsf{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\ell_{21} & 1 & 0 \\ -\ell_{31} & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathsf{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\ell_{32} & 1 \end{bmatrix}$$

   to a non-singular 3×3 matrix A are, i.e., what are the effects of $\mathsf{E}_1\mathsf{A}$ and $\mathsf{E}_2\mathsf{A}$?

   (b) Show that the inverse of the unit lower triangular matrix

$$\mathsf{L} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

   of the LU-factorization $\mathsf{A} = \mathsf{LU}$ is obtained as the product of elementary matrices, i.e., $\mathsf{L}^{-1} = \mathsf{E}_2\mathsf{E}_1$ by discussing how the system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ can be converted to $\mathsf{L}^{-1}\mathsf{A}\boldsymbol{x} = \mathsf{L}^{-1}\boldsymbol{b}$ such that $\mathsf{L}^{-1}\mathsf{A} = \mathsf{U}$, an upper triangular matrix.

   (c) What are the inverses of E_1 and E_2 given in (a)?

   (d) Based on the above discussion, how can one obtain the inverse of a unit lower triangular matrix?

5. Given

$$\mathsf{A} = \begin{bmatrix} 3 & 2 & -1 \\ 5 & 3 & 2 \\ -1 & 1 & -3 \end{bmatrix}, \quad \mathsf{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{5}{3} & 1 & 0 \\ -8 & 5 & 1 \end{bmatrix}, \quad \mathsf{U} = \begin{bmatrix} 3 & 2 & -1 \\ 0 & -\frac{1}{3} & \frac{11}{3} \\ 0 & 0 & 15 \end{bmatrix},$$

   obtain the inverse of A by solving $\mathsf{U}\mathsf{X}(:, j) = \mathsf{L}^{-1}\mathsf{I}(:, j)$ for $j = 1, 2, 3$.