Illinois Institute of Technology
Department of Applied Mathematics

Math 100: Introduction to profession

Fibonacci numbers via MATLAB computations

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Office Hours: Thursday, 3:30-4:30pm, E1 Room 125B,

Team: 2-3 (at most 5) members

Abstract

The classical Fibonacci numbers are defined as a recurrent sequence of integers such that each number, starting with third one, is equal to the sum of the previous two numbers (the first two numbers are ones). The goal of this project is to understand the mathematics behind this famous sequence, to extend the ideas to more general recurrent sequences and to implement in MATLAB computation of various recurrent sequences.

Project Plan

For the major part of the project the main reference will be Chapter 2 from *Experiments with* MATLAB by Cleve Moler `http://www.mathworks.com/moler/exm/chapters/fibonacci.pdf`, refereed in what follows as [M]. Also, you can google it (and get 328,000 hits), but please use the internet wisely and make proper citations to the original sources. References to Wiki are not allowed in this project, but Wiki can serve as a good starting point.

The special sign ♡ throughout the project indicates that the corresponding question/problem has to be answered in your final written part of the project.

1) History of Fibonacci numbers. Leonardo of Pisa, known as Fibonacci (a contraction of filius Bonaccio, "son of Bonaccio"), posted the following problem in his 1202 book *Liber Abaci*: A man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

Denote by f_n the number of pairs of rabbits after n months.

♡ Q1) Show that the sequence f_n satisfies the following relationship

$$f_{n+2} = f_{n+1} + f_n, \quad n = 1, 2, 3 \ldots, \tag{1}$$

with $f_1 = 1, f_2 = 2$.

♡ Q2) Why does the problem become less interesting (not to say obvious) if Fibonacci had not specified a month for the newborn pair to mature.

♡ Q3) Show another example where Fibonacci numbers are coming into play (you may use any source as a reference).

2) Reproducing Fibonacci sequence in MATLAB

In this part you should learn how to create/build your own function in MATLAB, how to use 'for loops', what is a recurrent function, how to plot and interpret graphs in MATLAB

♡ Q4) Write your MATLAB code that will reproduces the Fibonacci numbers. This becomes easy as long as you understand how 'for loop' works and what is the definition of the recurrent sequence called Fibonacci Numbers. Call the new function `myFibonacci`. It is up to you to choose what are the inputs and what are the outputs, however you have to write help for this function. Thus, running `help myFibonacci`, the user should get standard information about your function. Run this function for various inputs.
hint: a version of this function is at the end of this document and also in [7], but first try yourself to build one.

♡ Q5) Using the above function, plot the graph of Fibonacci numbers $\{(n, f_n),\ n = 1, 2, \ldots, 15\}$, and also do the log-plot $\{(n, \ln(f_n)),\ n = 1, 2, \ldots, 15\}$. Compare these two plots, and argue that Fibonacci numbers behave as a power function. Do Exercise 2.3 from the end of Chapter 2 from Moler.

♡ Q6) There is another way to find Fibonacci numbers by MATLAB. The key word is 'recurrent'. Read p.3 and p.4, Chapter 2, from Moler's book (function `fibnum`). Implement this function and compare the results with the function from Q4).

♡ Q7) Using `tic/toc` find what function is faster. Do Exercise 2.4 from the end of Chapter 2 from Moler.

3) General form for f_n. In fact, one can show that the general formula for the n-th term in the Fibonacci sequence is given by

$$f_{n-1} = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}(1-\phi)^n, \tag{2}$$

where $\phi = (1+\sqrt{5})/2$.

♡ Q8) The number ϕ is called the golden ratio. Why this ratio is called Golden?

♡ Q9) by direct substitution, show that f_n's defined as in (2) satisfy relation (1). This will imply that this is a (one) solution. Argue that the solution is unique.

Generally speaking, the existence and uniqueness of the solution of a given equation is a very fundamental mathematical question, and in many cases it is a non-obvious question. However, similar to Fibonacci numbers, there are classes of equation for which existence and uniqueness is an obvious question.

♡ Q10) What would make the Fibonacci sequence NOT to be unique, i.e. what would you change in the original problem that will yield to a non unique solution but the relationship (1) is preserved.

♡ Q11) Read p.7 [M] about how (2) is derived. Do Exercise 2.2 from [M].

4) Second order difference equations. This part of the project is subject to team size and overall progress. In this part we will deal with functional equations similar to Fibonacci sequence. A functional equation is an equation in which the unknown is a function. To solve a functional equation means to find all functions that satisfy a given relationship (equation). For example (1) together with $f_1 = 1, f_2 = 2$ is a functional equation, since the unknown sequence is a function $f : \mathbb{N} \to \mathbb{R}$. We solved this equation basically by guessing the solution and then verifying that this is a unique solution for this equation. In more general setup one should find the solution or even better to describe a general method of finding the solution for a certain class of equations. Again, this is a very general mathematical question, not to say that actually it is impossible to find an universal method of solving any functional equation. Soon you will have to take *Math252, Introduction do differential equations*, which will be entirely dedicated to solving functional equations of a specific type - differential equations.

Back to Fibonacci sequence. Assume that $g : \mathbb{N} \to \mathbb{R}$ is a function that satisfies the following relation

$$ag(n+2) + bg(n+1) + cg(n) = 0, \quad n \in \mathbb{N}, \tag{3}$$

where a, b, c are some given real numbers. Equation (3) is called (homogeneous) finite difference equation of second order. Any function g that satisfies (3) is called a solution. The problem is to find all solutions of this equation.

♡ Why Fibonacci sequence is a "particular case" of this equation? Well, almost particular case.

♡ Does this equation have a unique solution? If not, argue why?

♡ Give some concrete example of second order finite difference equations with obvious solutions.

The key point in solving (3) is to look at some "right" functions that may be solutions. We take $g(n) = \lambda^n, \ n \in \mathbb{N}$.

♡ Substitute $g(n) = \lambda^n$ in the equation of interest and find all $\lambda \in \mathbb{R}$ for which g is a solution. Describe all possible scenarios as number of distinct λ's.

Depending on how many λ's we have different type of solutions. However, regardless how complicated is the original equation we will be able to write down explicitly ANY solution, and yes, there are many, many solutions. Speaking mathematically we will be able to give the general solution of any second order difference equation.

```
 % Fibonacci sequence
% f = myFibonacci(n) generates the first n Fibonacci numbers.
f = zeros(n,1);
f(1) = 1;
f(2) = 2;
for k = 3:n
f(k) = f(k-1) + f(k-2);
end
```