Math 477: Midterm — Tuesday, Oct. 17, 2006

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1. Specify a 4×4 permutation matrix P such that left-multiplication of $\boldsymbol{x}$ by P reverses the order of the components of any vector $\boldsymbol{x} \in \mathbb{R}^4$.

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

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2. Under what circumstances does a small residual vector $\boldsymbol{r} = \boldsymbol{b} - A\boldsymbol{x}$ imply that $\boldsymbol{x}$ is an accurate solution to the linear system $A\boldsymbol{x} = \boldsymbol{b}$? State a formula that shows the connection between $\boldsymbol{r}$ and the error in the solution.

$$\frac{1}{K(A)} \frac{\|r\|}{\|b\|} \leq \frac{\|x - \tilde{x}\|}{\|x\|} \leq K(A) \frac{\|r\|}{\|b\|}$$

If $K(A)$ close to 1, then small r will ensure small error.

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3. Give an example of a 3×3 matrix A, other than the identity matrix, such that $\mathrm{cond}(A) = 1$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ works.

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4. Assuming you have the LU factorization of A available, write an algorithm to solve the equation $\boldsymbol{x}^* A = \boldsymbol{b}^*$ for $\boldsymbol{x}$.

First, note that $x^* A = b^* \iff A^* x = b$

Now, since $A = LU \iff A^* = U^* L^*$, we have

$x^* A = b^* \iff U^* L^* x = b$.

Algorithm:

1) Solve $U^* y = b$ for y (lower triang.)

2) Solve $L^* x = y$ for x (upper triang.)

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5. Assume L is an $m \times m$ matrix, and $\boldsymbol{b}$ an m-vector. Perform an operation count for the following algorithm:

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function x = Forwardsub(L, b)
for j = 1 : m
  x(j) = b(j)/L(j,j)
  for i = j+1 : m
    b(i) = b(i) - L(i,j)x(j)
  end
end
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$$\sum_{j=1}^{m} \left(1 + \sum_{i=j+1}^{m} 2\right) = \sum_{j=1}^{m} \left[1 + 2(m-j)\right]$$

$$= m + 2 \sum_{j=1}^{m} (m-j)$$

$$= m + 2 \Big[\underbrace{m^2 - \frac{m(m+1)}{2}}_{= \frac{m^2}{2} - \frac{m}{2}} \Big] = \underline{\underline{m^2}}$$

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6. Show how LU factorization with partial (row) pivoting works for the matrix

$$A = \begin{bmatrix} 4 & 7 & 3 \\ 1 & 3 & 2 \\ 2 & -4 & -1 \end{bmatrix}.$$

Be sure to give the final P, L, and U matrices.

First permutation $P_1 = I$

Then $L_1 P_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 7 & 3 \\ 1 & 3 & 2 \\ 2 & -4 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 3 \\ 0 & 5/4 & 5/4 \\ 0 & -15/2 & -5/2 \end{bmatrix}$

Now, $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (swap rows 2 and 3)

and

$$L_2 P_2 L_1 P_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/6 & 1 \end{bmatrix} \begin{bmatrix} 4 & 7 & 3 \\ 0 & -15/2 & -5/2 \\ 0 & 5/4 & 5/4 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & 7 & 3 \\ 0 & -15/2 & -5/2 \\ 0 & 0 & 5/6 \end{bmatrix}}_{=U}$$

Put together:

$$P = P_2 P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and $L = (L_2' L_1')^{-1}$ with $L_2' = L_2$ and

$$L_1' = P_2 L_1 P_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 0 & 1 \\ -1/4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/4 & 0 & 1 \end{bmatrix}$$

So $PA = LU \iff \boxed{\begin{bmatrix} 4 & 7 & 3 \\ 2 & -4 & -1 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & -1/6 & 1 \end{bmatrix} \begin{bmatrix} 4 & 7 & 3 \\ 0 & -15/2 & -5/2 \\ 0 & 0 & 5/6 \end{bmatrix}}$

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7. For the matrix

$$A = \begin{bmatrix} 3 & 4 \\ 0 & 1 \\ 4 & 0 \end{bmatrix}$$

(a) Find the Householder reflector $F = I - 2P$. that produces zeros in the first column of A.

(b) Apply F to A.

(c) How are F and FA related to the QR factorization of A?

(a) $F = I - 2P = I - 2vv^*$

where v is normalized $x + \text{sign}(x(1))\|x\|_2 e_1$

Here $x = a_1 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$ with $\|x\|_2 = \sqrt{9+16} = 5$

so that $v = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}$ and $\|v\| = \sqrt{64+16} = 4\sqrt{5}$

$\Rightarrow$ $v = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

and $\underline{\underline{F}} = I - 2vv^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$$= \underline{\underline{\begin{bmatrix} -3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \end{bmatrix}}}$$

(b) $\underline{\underline{FA}} = \underline{\underline{\begin{bmatrix} -5 & -12/5 \\ 0 & 1 \\ 0 & -16/5 \end{bmatrix}}}$

(c) $A = QR$ is given by

$\underline{\underline{A = Q_2^T Q_1^T R_1 R_2}}$ with $\underline{Q_1 = F}$
$\underline{R_1 = FA}$

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8. Consider the matrix

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}.$$

(a) Determine an SVD $A = U\Sigma V^T$ of A.

(b) Find A^{-1} using the SVD computed in (a).

(a) $A^TA = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$ with eigenvalues $\lambda_1 = 8$, $\lambda_2 = 2$

so that $\sigma_1 = \sqrt{8} = 2\sqrt{2}$, $\sigma_2 = \sqrt{2}$

The corresponding eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now $u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{8}}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Thus $\underline{A = U\Sigma V^T} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\underline{A^{-1}} = V\Sigma^{-1}U^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{8}} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 1/4 & 1/4 \\ -1/2 & 1/2 \end{bmatrix}$