Math 477 — Homework Assignment 4, due Oct.26, 2006

1. Given $A \in \mathbb{C}^{m\times n}$ of rank n and $\boldsymbol{b} \in \mathbb{C}^m$, consider the block 2×2 system of equations

$$\begin{bmatrix} I & A \\ A^* & O \end{bmatrix} \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix},$$

where I is the $m \times m$ identity matrix. Show that this system has a unique solution $[\boldsymbol{r}, \boldsymbol{x}]^T$, and that the vectors $\boldsymbol{r}$ and $\boldsymbol{x}$ are the residual and the solution of the least squares problem:

> Given $A \in \mathbb{C}^{m\times n}$ of full rank, $m \geq n$, $\boldsymbol{b} \in \mathbb{C}^m$, find $\boldsymbol{x} \in \mathbb{C}^n$ such that $\|\boldsymbol{b} - A\boldsymbol{x}\|$ is minimized.

2. Here is a stripped-down version of one of MATLAB's built-in m-files.

```
[U,S,V] = svd(A);
S = diag(S);
tol = max(size(A))*S(1)*eps;
r = sum(S > tol);
S = diag(ones(r,1)./S(1:r));
X = V(:,1:r)*S*U(:,1:r)';
```

Explain line-by-line what this code does. What is `X`?

3. Suppose an $m \times m$ matrix A is written in the block form $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{11} is $n \times n$ and A_{22} is $(m-n) \times (m-n)$. Assume that A is such that its LU factorization exists. Verify the formula

$$\begin{bmatrix} I & O \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

for "elimination" of the block A_{21}. The matrix $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is known as the *Schur complement* of A_{11} in A.

4. Let A be the 4×4 matrix

$$A = \begin{bmatrix} -1 & 1 & 0 & -3 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 1 & 2 \end{bmatrix}.$$

   (a) Compute the LU factorization of A with and without partial pivoting.

   (b) Determine $\det(A)$ from the 2 LU factorizations of A obtained in (a).

   (c) Describe how Gaussian elimination with partial pivoting can be used to find the determinant of a general square matrix.

5. Given a nonsingular matrix A. Describe how to find the inverse of A from its LU factorization $A = LU$ without explicitly computing inverse matrices.

6. Let A be a nonsingular square matrix and let $A = QR$ and $A^*A = U^*U$ be QR and Cholesky factorizations, respectively, with the usual normalizations $r_{jj}, u_{jj} > 0$. Is it true or false that $R = U$? Explain.

7. Give an example of a symmetric positive matrix that is not positive definite, i.e., construct a (2×2) matrix A with all positive entries such that $\boldsymbol{x}^T A\boldsymbol{x}$ is sometimes negative.