Math 477 — Homework Assignment 1, due Sept.14, 2006

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1. Let B be a 4×4 matrix to which we apply the following operations:

   (i) double column 1,

   (ii) halve row 3,

   (iii) add row 3 to row 1,

   (iv) interchange columns 1 and 4,

   (v) subtract row 2 from each of the other rows,

   (vi) replace column 4 by column 3,

   (vii) delete column 1 (so that the column dimension is reduced by 1).

   (a) Write the result as a product of eight matrices.

   (b) Write it again as a product ABC (same B) of three matrices.

2. The Pythagorean theorem asserts that for a set of n orthogonal vectors $\{\boldsymbol{x}_i\}$,

$$\left\| \sum_{i=1}^{n} \boldsymbol{x}_i \right\|^2 = \sum_{i=1}^{n} \|\boldsymbol{x}_i\|^2.$$

   (a) Prove this in the case $n = 2$ by an explicit computation of $\|\boldsymbol{x}_1 + \boldsymbol{x}_2\|^2$.

   (b) Show that this computation also establishes the general case, by induction.

3. Let $A \in \mathbb{C}^{m \times m}$ be Hermitian. An eigenvector of A is a nonzero vector $x \in \mathbb{C}^m$ such that $A\boldsymbol{x} = \lambda \boldsymbol{x}$ for some $\lambda \in \mathbb{C}$, the corresponding eigenvalue.

   (a) Prove that all eigenvalues of A are real.

   (b) Prove that if $\boldsymbol{x}$ and $\boldsymbol{y}$ are eigenvectors corresponding to distinct eigenvalues, then $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal.

4. What can be said about the eigenvalues of a unitary matrix?

5. Read Section 1.4 in the classnotes (Sections 2.1 and 2.2 in Kincaid/Cheney or Lecture 13 in Trefethen/Bau contain similar information).

6. If $\frac{1}{10}$ is correctly rounded to the normalized binary number $(1.a_1 a_2 \ldots a_{23})_2 \times 2^m$, what is the roundoff error? What is the relative roundoff error?

7. Give examples of real numbers x and y for which $fl(x \odot y) \neq fl(fl(x) \odot fl(y))$. Illustrate all four arithmetic operations using a machine with five decimal digits.

8. Consider the function $f(x) = x - \sin x$. Since $x \approx \sin x$ for small values of x, evaluation of f for such x involves a loss of significance. This loss of significance can be avoided by using the Taylor series expansion of $\sin x$. By using the error term of the Taylor expansion, show that at least seven terms are required if the error is not to exceed 10^{-9}.

9. Use Theorem 1.13 in the notes to estimate how many bits of precision are lost in a computer when we carry out the subtraction $x - \sin x$ for $x = \frac{1}{2}$?

10. In solving the quadratic equation $ax^2 + bx + c = 0$ by use of the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

there is a loss of significance when $4ac$ is small relative to b^2 because then

$$\sqrt{b^2 - 4ac} \approx |b|.$$

Suggest a method to circumvent this difficulty.