Math 477/577 — Computer Assignment 5, due Nov.9, 2006

1. Let A be a 10×10 random matrix with entries from the standard normal distribution, minus twice the identity. Write a program to plot $\|e^{tA}\|_2$ against t for $0 \le t \le 20$ on a log scale, comparing the result to the straight line $e^{t\alpha(A)}$, where $\alpha(A) = \max_j \Re(\lambda_j)$ is the *spectral abscissa* of A. Run the program for ten random matrices A and comment on the results. What property of a matrix leads to a $\|e^{tA}\|_2$ curve that remains oscillatory as $t \to \infty$?

   Hints: Recall that the matrix exponential is defined as

   $$e^{tA} = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}.$$

   Use `randn` and `expm` to deal with normally distributed random numbers and matrix exponentials, respectively.

2. Let A be the 32×32 matrix with -1 on the main diagonal, 1 on the first and second superdiagonals, and 0 elsewhere.

   For $A \in \mathbb{C}^{m \times m}$ with spectrum $\Lambda(A) \subseteq \mathbb{C}$ and $\varepsilon > 0$, we define the 2-norm *ε-pseudospectrum of A*, $\Lambda_\varepsilon(A)$, to be the set of numbers $z \in \mathbb{C}$ satisfying any of the following conditions:

   (i) z is an eigenvalue of $A + \delta A$ for some δA with $\|\delta A\|_2 \le \varepsilon$;
   (ii) There exists a vector $\boldsymbol{u} \in \mathbb{C}^m$ with $\|(A - zI)\boldsymbol{u}\|_2 \le \varepsilon$ and $\|\boldsymbol{u}\|_2 = 1$;
   (iii) $\sigma_m(zI - A) \le \varepsilon$;
   (iv) $\|(zI - A)^{-1}\|_2 \ge \varepsilon^{-1}$.

   The matrix $(zI - A)^{-1}$ in (iv) is known as the *resolvent* of A at z; if z is an eigenvalue of A, we use the convention $\|(zI - A)^{-1}\|_2 = \infty$. In (iii), σ_m denotes the smallest singular value of A.

   (a) Using an SVD algorithm built into MATLAB together with MATLAB's `contour` command, generate a plot of the boundaries of the 2-norm ε-pseudospectra of A for $\varepsilon = 10^{-1}, 10^{-2}, \ldots, 10^{-8}$.
   (b) Produce a `semilogy` plot of $\|e^{tA}\|_2$ against t for $0 \le t \le 50$. What is the initial growth rate of the curve before the eventual decay sets in? Can you relate this to your plot of pseudospectra? (Compare to the previous problem.)