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MATH 350 — Midterm 1, February 17, 2011

Make sure to show all your work!

1. The k^{th}-order derivative of the function $f(x) = \ln(x+1)$ is given by $\frac{d^k}{dx^k}\ln(x+1) = \frac{(-1)^{k-1}(k-1)!}{(x+1)^k}$, $k = 1, 2, \ldots$.

8 (a) What is the Taylor series for $f(x)$ about the point $x_0 = 0$?

8 (b) How many terms are needed to compute $\ln(2)$ based on the series from (a) with an error of at most 10^{-6}?

(a) Taylor series: $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

Using $f(x) = \ln(x+1)$ and $x_0 = 0$ so that $f(x_0) = \ln(1) = 0$ we have

$$\underline{\ln(x+1)} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{(0+1)^k\, k!}(x-0)^k = \underline{\sum_{k=1}^{\infty} (-1)^{k-1}\frac{x^k}{k}} \qquad (*)$$

(b) For the error we know

$$|E_{n+1}(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \right|$$

using $x_0 = 0$ and the given derivative formula

$$|E_{n+1}(x)| = \left| \frac{(-1)^n\, n!}{(n+1)!\,(\xi+1)^{n+1}} x^{n+1} \right| = \frac{1}{n+1}\left| \left(\frac{x}{\xi+1}\right)^{n+1} \right|$$

We want error for $\ln(2) = \ln(1+1)$, so $x = 1$ and $0 < \xi < 1$.

Therefore, $|E_{n+1}(1)| = \frac{1}{n+1}\left|\frac{1}{(\xi+1)^{n+1}}\right| < \frac{1}{n+1}$

To have this error no larger than 10^{-6} we take n such that

$\frac{1}{n+1} < 10^{-6} \iff n+1 > 10^6$, so <u>at least 1 000 001 terms</u>.

The estimate also follows with the alt. series test for $\ln(2) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$ (just use $x=1$ in $(*)$)

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2. Consider the expression $x\left(\sqrt{x+1}-\sqrt{x}\right)$ which becomes problematic to evaluate for large values of x. For example, for $x = 10^8$ MATLAB produces an answer of 5000.000055588316 while we should have (using the same number of 16 significant digits) 4999.999987500000.

4 (a) Why is the MATLAB answer not more accurate?

4 (b) Find a mathematically equivalent form of the expression above that permits a more accurate evaluation and use your calculator to evaluate this new expression at $x = 10^8$.

4 (c) Repeat part (b) for the expression $\frac{1-\cos x}{x^2}$ and $x = 10^{-8}$. The correct answer, rounded to 16 significant digits, is 0.5000000000000000.

4 (d) What is the correct value of the expression from part (c) for $x = 10^{-8}$ if 20 significant digits are revealed? Show the work that reveals these additional digits.

Note: depending on how fancy your calculator is, it may provide enough precision to already evaluate the original expressions above "correctly". This does **not** give you an excuse not to perform all the work requested for this problem ☺.

(a) We have loss of significant digits because $\sqrt{10^8+1}$ is very close to $\sqrt{10^8}$.

(b) $x\left(\sqrt{x+1}-\sqrt{x}\right) = x\left(\sqrt{x+1}-\sqrt{x}\right)\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} = x\,\frac{x+1-x}{\sqrt{x+1}+\sqrt{x}} = \frac{x}{\sqrt{x+1}+\sqrt{x}}$

For $x = 10^8$ MATLAB now gives 4999.9999874999

(c) Since $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

we have $\frac{1-\cos x}{x^2} = \frac{1}{2} - \frac{x^2}{4!} + \frac{x^4}{6!} - \cdots$

For $x = 10^{-8}$ MATLAB gives us 0.5000000000000000.

(d) To get more accuracy, compute $\frac{x^2}{4!} = \frac{10^{-16}}{4!}$ first and then subtract manually from 0.5. This gives

$0.5 - 4.1666666666666667 \times 10^{-18}$

$= 0.\underbrace{4999999999999999}_{16}583$

(18)

3. You may use exact arithmetic for this problem. Consider the matrices

$$\mathsf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}, \qquad \mathsf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}, \qquad \mathsf{U} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

4 (a) Show that an LU-decomposition of A using an appropriate pivoting strategy results in the factors L and U listed above. Show and explain all steps of your work.

6 (b) Use the matrices L and U from above to efficiently compute the inverse of A.

4 (c) What is the ℓ_∞-condition number $\kappa_\infty(\mathsf{A})$ of the matrix A given above?

4 (d) What can you say about the relative error of the solution you've computed for a linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ with A as above assuming you've checked that your computed solution has a residual of $\|\boldsymbol{r}\|_\infty = 10^{-8}\|\boldsymbol{b}\|_\infty$?

(a) $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix} \xrightarrow[(3)-(-2)(1)]{(2)-(1)} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 3 & -1 \end{bmatrix} \xrightarrow{(3)-3(2)} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} = U$

multipliers end up in $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}$, no pivoting was used

(b) Find A^{-1} by solving $LUX = I$ in two steps: $\begin{cases} LY = I \\ UX = Y \end{cases}$

$LY = I \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix} Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xRightarrow{\text{forward subst.}} Y = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$

$UX = Y \Leftrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} X = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix} \xRightarrow{\text{back subst.}} X = \begin{bmatrix} 2 & -1 & 0 \\ 3/2 & -1/2 & 1/2 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} = A^{-1}$

(c) $\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$

with $\|A\|_\infty = 5$, $\|A^{-1}\|_\infty = 9/2$ (using max. of row sums)

$\Rightarrow \kappa_\infty(A) = 45/2$

(d) We know $\frac{1}{\kappa(A)} \frac{\|r\|_\infty}{\|b\|_\infty} \le \frac{\|x - x^*\|_\infty}{\|x\|_\infty} \le \kappa(A) \frac{\|r\|_\infty}{\|b\|_\infty}$

Here: $4.4 \times 10^{-10} = \frac{2}{45} 10^{-8} \le \frac{\|x - x^*\|_\infty}{\|x\|_\infty} \le \frac{45}{2} 10^{-8} = 2.25 \times 10^{-7}$

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4. (a) Find the quadratic polynomial that interpolates the data

x	0	1	2
y	-1	-1	7

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Simplify your answer to the standard form $p(x) = ax^2 + bx + c$ with appropriate values of a, b and c.

(b) What does the degree five Lagrange basis function L_4 look like for interpolation of

x	-2	0	1	2	4	7
y	2.1	3.5	-1.8	3.5	0	0

?

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Provide both a formula for $L_4(x)$ and a (very) rough sketch of the graph of L_4 over the interval $[-2, 7]$. Be sure to label your axes appropriately.

(a) $$\underline{p(x)} = \frac{(x-1)(x-2)}{(-1)(-2)}(-1) + \frac{(x-0)(x-2)}{1(-1)}(-1) + \frac{(x-0)(x-1)}{(2)(1)}(7)$$

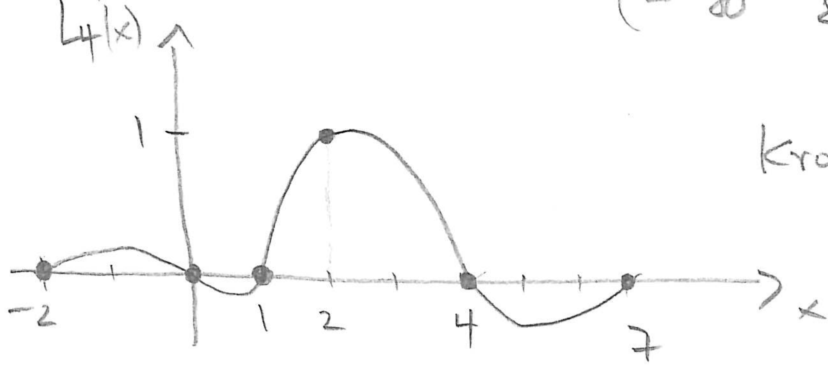
$$= -\frac{1}{2}(x-1)(x-2) + x(x-2) + \frac{7}{2}x(x-1)$$

$$= \underline{4x^2 - 4x - 1}$$

(b) $$\underline{L_4(x)} = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_5)(x-x_6)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)(x_4-x_6)}$$

$$= \frac{(x+2)\,x\,(x-1)(x-4)(x-7)}{(4)(2)(1)(-2)(-5)} = \underline{\frac{(x+2)\,x\,(x-1)(x-4)(x-7)}{80}}$$

$$\left(= \frac{x^5}{80} - \frac{x^4}{8} + \frac{3x^3}{16} + \frac{5x^2}{8} - \frac{7x}{10}\right)$$



Kronecker-delta property

$L_4(x_4) = L_4(2) = 1$

and zero at other nodes

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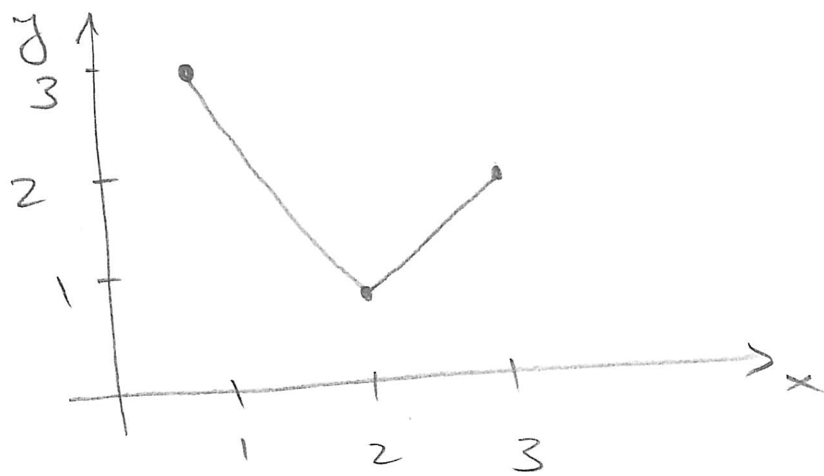
5. Consider the data

x	1	2	3
y	3	1	2

and provide both a formula for and the graph of the piecewise linear interpolant of this data over the interval $[1, 3]$.

$$\ell(x) = \begin{cases} \ell_1(x) = 3 + \frac{1-3}{2-1}(x-1) = 5-2x, & 1 \le x \le 2 \\ \ell_2(x) = 1 + \frac{2-1}{3-2}(x-2) = x-1, & 2 < x \le 3 \end{cases}$$



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6. What is the result of the following sequence of MATLAB commands:

```
A = [6 -2 2 4;
     12 -8 6 10;
     3 -13 9 3;
     -6 4 1 -18];
b = [12; 34; 27; -38];
[n,n] = size(A);
p = (1:n)';
for k = 1:n-1
    [r,m] = max(abs(A(k:n,k)))        % Output here
    m = m+k-1                          % Output here
    if (A(m,k) ~= 0)
       if (m ~= k)
          A([k m],:) = A([m k],:)    % Output here
          p([k m]) = p([m k])        % Output here
       end
    end
end
b(p)                                   % Output here
```

Note that this code contains an excerpt from `lutx.m`. However, due to the fact that it has been removed from its context it probably behaves differently than you might expect. Therefore, *follow the code carefully.* It produces quite a bit of output. Please list all of it.

MATLAB Help states

> C = max(A) returns the largest elements along different dimensions of an array.
> If A is a vector, max(A) returns the largest element in A.
> If A is a matrix, max(A) treats the columns of A as vectors, returning a row vector containing the maximum element from each column.
> [C,I] = max(...) finds the indices of the maximum values of A, and returns them in output vector I. If there are several identical maximum values, the index of the first one found is returned.

Output:

k=1

$[r, m] = [12, 2]$

$m = 2$

$$A = \begin{bmatrix} 12 & -8 & 6 & 10 \\ 6 & -2 & 2 & 4 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix}$$

$$p = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

k=2

$[r, m] = [13, 2]$

$m = 3$

$$A = \begin{bmatrix} 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ 6 & -2 & 2 & 4 \\ -6 & 4 & 1 & -18 \end{bmatrix}$$

$$p = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}$$

k=3

$[r, m] = [2, 1]$

$m = 3$

SKIPPED by code

$$b(p) = \begin{bmatrix} 34 \\ 27 \\ 12 \\ -38 \end{bmatrix}$$