Math 350 — Homework Assignment 7, due April 21, 2011

1. (a) Use two appropriately chosen Taylor expansions to obtain the error term for the non-symmetric derivative approximation formula

$$f'(x) \approx \frac{f(x+3h) - f(x-h)}{4h}.$$

   (b) What is wrong with the following argument? If we add the following two Taylor expansions

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(\xi) \\ f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(\xi) \end{aligned}$$

   then we obtain the *exact* derivative formula

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

2. In the slides I claimed that we can obtain a formula to approximate the k^{th}-order derivative of some unknown function f if we perform the following steps:

   - Construct the degree $n-1$ (with $n > k$) Lagrange interpolating polynomial p to f at a *generic* set of points $x_1, x_2, \ldots, x_n$, i.e.,

$$p(x) = \sum_{i=1}^{n} f(x_i) \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}.$$

   - Differentiate p k times.
   - Replace $x_1, x_2, \ldots, x_n$ in the formula for $p^{(k)}(x)$ by a *specific* set of points chosen relative to x.

   Follow the above procedure with $n = 3$ (quadratic polynomial), $k = 2$ (second-order derivative) and $x_1 = x - h$, $x_2 = x$ and $x_3 = x + h$ in the third step to obtain the formula

$$D_h^{(2)} f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

   for approximating $f''(x)$.

3. If you have a numerical method that satisfies the following error estimate

$$F_h = F - k_1 h^{1/2} - k_2 h - k_3 h^{3/2} - \ldots,$$

   then how should the Richardson combination of F_h and $F_{\frac{h}{2}}$ be formed so that you obtain improved accuracy? How accurate is this combination?

4. Rewrite the following initial value problems so that they can be solved with one of the initial-value problem solvers we studied:

   (a) $y(t) + 2y(t)y'(t) - y'(t) = 0$, $y(t_0) = y_0$,

   (b) $\log y'(t) = t^2 - y^2(t)$, $y(t_0) = y_0$,

   (c) $(y'(t))^2(1 - t^2) = y(t)$, $y(t_0) = y_0$.

(d) $y'''(t) = t + y(t) + 2y'(t) + 3y''(t)$, $y(1) = 3$, $y'(1) = -7$, $y''(1) = 4$.

(e) The system of ODEs

$$
\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) - 2x(t)z(t)\frac{\mathrm{d}}{\mathrm{d}t}x(t) &= 3x^2(t)y(t)t^2 \\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) - e^{y(t)}\frac{\mathrm{d}}{\mathrm{d}t}y(t) &= 4x(t)t^2z(t) \\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t) - 2t\frac{\mathrm{d}}{\mathrm{d}t}z(t) &= 2te^{x(t)z(t)} \\
x(1) = y(1) &= z'(1) = 3, \quad x''(1) = y'(1) = z(1) = 2.
\end{aligned}
$$

5. Solve (by hand) the differential equation

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}y(t) &= -ty^2(t) \\
y(0) &= 2
\end{aligned}
$$

at $t = -0.2$ using one step of the classical second-order Runge-Kutta method.